5 Some Fundamentals of Graph Theory

The purpose of this section is to establish some basic terminologies and concepts of graph theory, which are necessary for optimization problems on networks and graphs. For more details, see Ahuja et.al [1], Bondy and Murty [2], and West [3].

5.1 (Undirected) Graphs

A graph, denoted, $G = (V, E)$, is a pair $(V, E)$ where $V$ is a set of vertices and $E$ is a set of two-element subsets of $V$ called edges. ($E \subseteq \{(i, j) : i, j \in V\}$).

**Definitions and Terminologies**: Let $G = (V, E)$ be a given (undirected) graph.

- Let $u, v \in V$, where $u \neq v$, i.e. $u, v$ are distinct vertices in $V$. Then $u$ and $v$ are said to be **adjacent** if the edge $(u, v) \in E$.
- An edge $(i, j) \in E$ is said to be **incident** to the vertices $i$ and $j$.
- Let $v \in V$. An edge of the type $(v, v)$ is called a **loop**.
- Let $u, v \in V$ where $u \neq v$. Then the edges $(u, v)$ and $(u, v)$ are called **parallel edges**.
- The **degree** of a vertex is the number of edges incident to the vertex, with a loop counting twice.
- A graph $G$ is **simple** if it does NOT contain any loops nor parallel edges.

For examples of the aforementioned terms, refer to Figure 1 and Figure 2.

![Figure 1: An undirected graph $G = (V, E)$](image1)

![Figure 2: A simple undirected graph $G = (V, E)$](image2)
Let $G = (V, E)$ be a graph.

- Let $\bar{V} \subseteq V$. Then a subgraph of $G$ induced by $\bar{V}$ is the graph whose vertex set is $\bar{V}$ and whose edge set is $\bar{E} \equiv \{(i, j) \in E : i, j \in \bar{V}\}$.

- Let $\tilde{E} \subseteq E$. Then a subgraph of $G$ induced by $\tilde{E}$ is the graph whose edge set is $\tilde{E}$ and whose vertex set is $\tilde{V} \equiv \{i, j \in V : (i, j) \in \tilde{E}\}$.

- A spanning subgraph of $G$ is a subgraph of $G$ (either induced by a vertex or edge set) that contains all vertices of $G$.

For example, in the graph of Figure 2, the subgraph induced by the vertex set $\bar{V} = \{1, 2, 3, 4, 6, 8, 10\}$ is given in Figure 3. And the subgraph induced by the edge set $\tilde{E} = \{e_1, e_5, e_7, e_8, e_9, e_{11}\}$ is given in Figure 4.
5.2 Some Structures of Graphs

Let \( G = (V, E) \) be a graph.

Refer to Figure 5 for an example.

- A walk from vertex \( v_0 \) to vertex \( v_k \) is a finite sequence \( v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k \), where each \( v_i \in V \) for \( i = 0, 1, 2, \ldots, k \) and each \( e_i = (v_{i-1}, v_i) \in E \) for \( i = 1, 2, \ldots, k \).
  An example of a walk from vertex 4 to vertex 1 in Figure 5 is 4, f, 1, a, 1, b, 2, h, 3, l, 5, c, 2, h, 3, g, 1.

- A trail from vertex \( v_0 \) to vertex \( v_k \) is a walk from vertex \( v_0 \) to vertex \( v_k \) which contains NO repeated edges.
  An example of a trail from vertex 4 to vertex 1 in Figure 5 is 4, f, 1, a, 1, b, 2, h, 3, l, 5, d, 6, m, 3, g, 1.

- A path from vertex \( v_0 \) to vertex \( v_k \) is a trail from vertex \( v_0 \) to vertex \( v_k \) which contains NO repeated vertices.
  An example of a path from vertex 4 to vertex 1 in Figure 5 is 4, e, 6, m, 3, g, 1.

- A tour is a trail from vertex \( v_0 \) to vertex \( v_0 \).

- A cycle is a trail from vertex \( v_0 \) to vertex \( v_0 \) which contains NO other repeated vertices.
  An example of a cycle in Figure 5 is 1, b, 2, h, 3, m, 6, e, 4, f, 1.

A graph \( G = (V, E) \), is said to be connected if there exists a path from vertex \( u \) to vertex \( v \) for all pairs of vertices \( u, v \in V \).

Remark: Connectivity is an equivalence relation on the set of vertices \( V \). WHY?
In other words, for \( u, v, w \in V \), why are the following three conditions satisfied?
(i) \( u \) is connected to \( u \),
(ii) if \( u \) is connected to \( v \) then \( v \) is connected to \( u \),
(iii) if \( u \) is connected to \( v \) and if \( v \) is connected to \( w \), then \( u \) is connected to \( w \).

Corollary of Remark:
Given a graph \( G = (V, E) \), there exists a partition of the vertex set \( V \) into non-empty subsets, say, \( V_1, V_2, \ldots, V_\omega \), such that two vertices \( u \) and \( v \) are connected if and only if \( u \) and \( v \) belong to the same vertex set \( V_i \).

The subgraphs induced by \( V_1, V_2, \ldots, V_\omega \), i.e. \( G[V_1], G[V_2], \ldots, G[V_\omega] \) are called the components of \( G \). For examples of finding the components of a given graph \( G = (V, E) \) see Figure 6 and Figure 7.
Figure 6: $G = (V, E)$ has more than one component

Figure 7: $G = (V, E)$ has exactly one component

- A **forest** is a graph with no cycles.

- A **tree** is a forest that is connected.

- Let $G = (V, E)$ be a graph. A **spanning tree** of $G$ is a spanning subgraph of $G$ that is also a tree.

**Theorem** Let $G = (V, E)$ be a graph. Then $G$ is connected if and only if $G$ contains a spanning tree.

**Theorem** Let $T = (V, E)$ be a tree. Then $T$ has exactly one more vertex than it has edges, i.e., $|V| = |E| + 1$.

**Proof:** Use mathematical induction on $|V|$.
5.3 Directed Graphs

A directed graph, denoted, \( G = (V, A) \), is a pair \( (V, A) \) where \( V \) is a set of vertices and \( A \) is an ordered set of two-element subsets of \( V \) called arcs, \( A \subseteq \{(i, j): i, j \in V\} \), and note that the arcs \( (i, j) \neq (j, i) \).

For an example of a directed graph, refer to Figure 8.

![Figure 8: A directed graph, \( G = (V, A) \)](image)

Given a directed graph \( G = (V, A) \), the underlying graph associated with \( G \), is the undirected graph where the order on the arcs of \( A \) are ignored.

For instance, the undirected graph \( G = (V, E) \) in Figure 1 is the underlying graph of the directed graph \( G = (V, A) \) in Figure 8.

5.4 Some structures of Directed Graphs

Let \( G = (V, A) \) be a given directed graph.

- A walk from vertex \( v_0 \) to vertex \( v_k \) is a finite sequence \( v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k \), where each \( v_i \in V \) for \( i = 0, 1, 2, \ldots, k \) and each \( a_i \equiv (v_{i-1}, v_i) \in A \) or \( a_i \equiv (v_i, v_{i-1}) \in A \) for \( i = 1, 2, \ldots, k \).

- A directed walk from vertex \( v_0 \) to vertex \( v_k \) is a walk where each \( a_i \) is restricted to direction, i.e. \( a_i \equiv (v_{i-1}, v_i) \in A \).

- A trail from vertex \( v_0 \) to vertex \( v_k \) is a walk from vertex \( v_0 \) to vertex \( v_k \) which contains NO repeated arcs.

- A directed trail from vertex \( v_0 \) to vertex \( v_k \) is a directed walk from vertex \( v_0 \) to vertex \( v_k \) which contains NO repeated arcs.

- A path or a chain from vertex \( v_0 \) to vertex \( v_k \) is a trail from vertex \( v_0 \) to vertex \( v_k \) which contains NO repeated vertices.

- A directed path from vertex \( v_0 \) to vertex \( v_k \) is a directed trail from vertex \( v_0 \) to vertex \( v_k \) which contains NO repeated vertices.
• A cycle is a trail from vertex $v_0$ to vertex $v_0$ which contains NO other repeated vertices.

• A directed cycle or a circuit is a directed trail from vertex $v_0$ to vertex $v_0$ which contains NO other repeated vertices.

Let $G = (V, A)$ be a directed graph.

• Let $u, v \in V$. Then vertex $u$ is said to be reachable from vertex $v$ if there exists a directed path from $v$ to $u$ in $G$.

• $G$ is said to be connected if there exists a path from $v$ to $u$ for every pair of vertices $v, u \in V$.

• $G$ is said to be strongly connected if $u$ and $v$ are reachable from each other, for every pair of vertices $v, u \in V$.

Example of a connected but not strongly connected directed graph is the graph in Figure 8. Why is it not strongly connected?

Example of a strongly connected directed graph is given in Figure 9.

![Figure 9: A strongly connected directed graph, $G = (V, A)$](image)

Why is it strongly connected?
5.5 References

