

Math/Mgmt 3502 (Ng/Fall 2007)
 Queueing Handout 2
 November 27, 2007.

4.3 Stochastic, Counting, and the Poisson Processes.

Let X be a discrete random variable whose values are $0, 1, 2, \dots$. Then X is a *Poisson* random variable with parameter $\lambda > 0$ if

$$p(i) = P\{X = i\} = \frac{e^{-\lambda} \lambda^i}{i!} \quad \text{for } i = 0, 1, 2, \dots$$

A *stochastic process*, $\{X(t) : t \in T\}$ is a collection of random variables. That is, for each $t \in T$, $X(t)$ is a random variable. The index t is often interpreted as time and, as a result, we refer to $X(t)$ as the *state* of the process at time t . For example, $X(t)$ might be the total number of customers that have entered a supermarket by time t ; or the number of customers in the supermarket at time t ; or the total amount of sales that have been recorded in the market by time t .

The set T is called the *index* set of the process. If T is a countable set then the stochastic process is said to be a *discrete-time* process. If T is an interval of real line, the stochastic process is said to be a *continuous-time* process. For instance, $\{X_n : n = 0, 1, 2, \dots\}$ is a discrete-time stochastic process indexed by the non-negative integers; while $\{X(t) : t \geq 0\}$ is a continuous-time stochastic process indexed by the non-negative real numbers.

The *state space* of a stochastic process is defined as the set of all possible values that the random variables $X(t)$ can assume. Thus, a stochastic process can be thought of as a family (or collection) of random variables that describes the evolution through time of some (physical) process.

A stochastic process $\{N(t) : t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of “events” that have occurred up to time t .

To be pedantic, the stochastic process $\{N(t) : t \geq 0\}$ is a counting process if $N(t)$ satisfies the following properties:

- (i) $N(t) \geq 0$.
- (ii) $N(t)$ is integer valued.
- (iii) If $s < t$, then $N(s) \leq N(t)$.
- (iv) For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval (s, t) .

Some examples of counting processes are the following.

1. If we let $N(t)$ be the number of people who have entered a particular store at or prior to time t , then $\{N(t) : t \geq 0\}$ is a counting process in which an event corresponds to a person entering the store. (Note that if we had let $N(t)$ be the number of people in the store at time t , then $\{N(t) : t \geq 0\}$ would **not** be a counting process. (*Why not?*))
2. If we say that an event occurs whenever a child is born, then $\{N(t) : t \geq 0\}$ is a counting process when $N(t)$ equals the total number of people who were born by time t . (Does $N(t)$ include persons who have kicked the bucket by time t ? Explain why it must.)
3. If $N(t)$ is the number of goals that a given soccer player has scored by time t , then $\{N(t) : t \geq 0\}$ is a counting process. An event of this process will occur whenever the soccer player scores a goal.

A counting process is said to possess *independent increments* if the numbers of events which occur in disjoint time intervals are independent. For example this means that the number of events which have occurred by time 10 (i.e. $N(10)$ must be independent of the number of events occurring between times 10 and 15 (i.e. $N(15) - N(10)$).

Is the assumption of independent increments reasonable for example 1 above? Example 2? Example 3?

A counting process is said to possess *stationary increments* if the distribution of the number of events which occur in any time interval depends only on the length of the time interval. In other words, the process has stationary increments if the number of events in the interval $(t_1 + s, t_2 + s)$ (i.e. $N(t_2 + s) - N(t_1 + s)$) has the same distribution as the number of events in the interval (t_1, t_2) (i.e. $N(t_2) - N(t_1)$) for all $t_1 < t_2$, and $s > 0$.

Is the assumption of stationary increments reasonable for example 1 above? Example 2? Example 3?

One of the most important counting processes is the *Poisson process* which is defined as follows.

The counting process $\{N(t) : t \geq 0\}$ is said to be a *Poisson process having rate λ* , $\lambda > 0$, if

- (i) $N(0) = 0$.
- (ii) The process has independent increments.
- (iii) The number of events in any interval of length t is a Poisson random variable with mean λt . That is, for all $s, t \geq 0$,

$$P\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!} \quad \text{for } n = 0, 1, 2, \dots$$

Note that it follows from condition (iii) that a Poisson process has stationary increments and also that

$$E[N(t)] = \lambda t$$

which is why λ is called the *rate* of the process.