MATH 4452 (Ng/Spring 2012)
Assignment 4
due February 16, 2012.

1. (20pts.) Solve each of the following second-order, homogeneous, DDS with constant coefficients using the special methods we discussed in class.

   (a)
$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n = 2, 3, \ldots$$
$$a_0 = 2,\ a_1 = 8$$

   (b)
$$a_n = 8a_{n-1} - 16a_{n-2} \quad \text{for } n = 2, 3, \ldots$$
$$a_0 = 1,\ a_1 = 6$$

2. (20pts.) Recall the discrete model for the predator-prey problem as

$$O_n = \alpha O_{n-1} + \beta R_{n-1} \quad \text{for } n = 1, 2, \ldots$$
$$R_n = -pO_{n-1} + qR_{n-1} \quad \text{for } n = 1, 2, \ldots$$

   where

   O_n and R_n denote the population (in thousands) of *owls* and *rats*, respectively, after n years, and the parameters α, β, p, q are positive real numbers.

   Suppose
$$O_0 = 2\,,\ R_0 = 5, \quad \alpha = \frac{1}{3}, \beta = \frac{5}{4}, p = \frac{1}{9}, q = \frac{4}{3}$$

   (a) Solve the above DDS, analytically, using *eigenvalue* methods.
   (b) In the long run, what is the annual percentage yearly growth/death rate for these animals?
   (c) For a large enough time, n, what is the ratio of owls to rates?

3. (20pts.) Consider the DDS:
$$a_n = 0.85a_{n-1} + 0.8b_{n-1} \quad \text{for } n = 1, 2, \ldots$$
$$b_n = -0.15a_{n-1} + 0.9b_{n-1} \quad \text{for } n = 1, 2, \ldots$$

   Suppose, in the above DDS, we have
$$a_0 = 2\,,\ b_0 = 5$$

   (a) Generate, via Mathematica, a table of ordered pairs, (a_n, b_n) for $n = 1, 2, \ldots, N$ where N is large enough that you can see some pattern.
   (b) Plot the graph of b_n versus a_n, based on the numerical data you obtained in part (a).
   (c) Based on your numerical solutions above, what is the behaviour of a_n and b_n in the long run, i.e as $n \to \infty$.

   (Enclose your mathematica printout (identify which is which) with your written answers.)

4. (30pts.) Consider the Predator-Prey model (*continuous dynamical system*) that we did in class, i.e.:

$$\frac{dH}{dt} = (\alpha - \beta P)H$$
$$\frac{dP}{dt} = (-\mu + \phi H)P$$

   where $H(t)$ and $P(t)$ represent the populations of the host-prey and the parasite-predator, respectively, after time $t \geq 0$.

(a) Suppose

$$\alpha = 1.2,\ \beta = 0.02,\ \mu = 2.5,\ \phi = 0.05$$

what are the equilibrium points, (H_e, P_e)?

(b) Plot a few trajectories of the function relating H and P on the (H, P)-plane. (Use mathematica, and use appropriate initial values for H and P.)

(c) With your chosen values of initial values, H_0 and P_0, does it look like both predator and prey can co-exist peacefully? Explain.

(d) Using NDSolve to obtain the numerical solutions to the above rate equations, plot the trajectory of H versus time, t.

(e) Using NDSolve to obtain the numerical solutions to the above rate equations, plot the trajectory of P versus time, t.

(f) Based on your answers above, is there evidence for periodicity in the behaviors of the populations of both predator and prey? Explain.

5. (30pts.) (Crazy Lotka-Volterra model, from Handout 5.)

Rhett and Scarlett are two lovers who share and thrive on a *sadistic relationship*, which is typical of immature people.

The way their relationship evolves as time goes by can be described in the following paragraph.

> The more Rhett loves Scarlett, the more she begins to dislike him. On the other hand, when Rhett's love for Scarlett begins to taper off, the more her affection for him begins to grow. In other words, if the amount of Rhett's love is higher then the rate of growth of Scarlett's love is smaller.
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> For Rhett's part, his love for Scarlett grows when she loves him and dissipates when she dislikes him. In other words, if the amount of Scarlett's love is higher then the rate of growth of Rhett's love is also higher.

Let $R(t)$ be (the amount of) Rhett's love or hate for Scarlett at any time t; and let $S(t)$ be (the amount of) Scarlett's love or hate for Rhett at any time t. (*For either function, a positive value means love and a negative value means hate*).

As a simple case, let us assume that *the rate of change of* Rhett*'s love or hate* has a linear relationship with *the amount of* Scarlett*'s love or hate*; and the former is zero when the latter is zero. And similarly, we assume that *the rate of change of* Scarlett*'s love or hate* also varies linearly with *the amount of* Rhett*'s love or hate*; and the former is zero when the latter is zero.

(a) Based on the above description, set up the equation that describes the *rate of change of* Rhett*'s love or hate for* Scarlett with respect to *time*, at any time t. (**Specify the condition on the constant of proportionality.**)

(b) Set up the equation that describes the *rate of change of* Scarlett*'s love or hate for* Rhett with respect to *time*, at any time t. (**Specify the condition on the constant of proportionality.**)

(c) Based on the two coupling differential equations you obtained in parts (a) and (b), set up the equation that describes the *rate of change of* Scarlett's love or hate *with respect to* Rhett's love or hate. (**Hint: Use the Chain Rule**).

(d) Based on your equation in part (c), solve for a direct relationship between S, (Scarlett's love/hate), in terms of R, (Rhett's love/hate). (In other words, come up with an equation model that will describe the relationship between S and R.)

(e) Based on your answer to part (d), sketch the graph of S versus R. Explain, clearly, why Scarlett and Rhett's relationship is a *never-ending* cycle of **love** and **hate**.