MATH 4452 (Ng/Spring 2012)
Assignment 3
due February 9, 2012.

1. (20pts.) Suppose Lake Mini-Morris contains 20 thousand gallons of water. On the average, each week two thousand gallons of water flows out of the lake, to be replaced by rain water. Originally, the lake contains 20 pounds of insecticides. Each thousand gallons of rain water flowing into Lake Mini-Morris contains 2 pounds of insecticide washed in from a nearby farmland.

   (a) Develop a discrete dynamical system (DDS) to describe the number of pounds of insecticide in Lake Mini-Morris after n weeks.
   (*Make sure you define your function or terms.*)

   (b) How many pounds of insecticide will be in Lake Mini-Morris at the end of the fourth week?

   (c) After a long period of time, how many pounds of insecticide will be in the lake?

   (d) Draw a *cobweb diagram* for this dynamical system.
   (Recall that the cobweb diagram is the figure obtained by connecting the sequence of ordered pairs: $\{(a_0, a_0), (a_0, a_1), (a_1, a_1), (a_1, a_2), (a_2, a_2), \ldots\}$.)

2. (10pts.) For what values of r will the following (DDS) have an equilibrium value at $x = 0.5$?

$$a_n = ra_{n-1} - 2$$

   (Recall that an equilibrium value of a DDS, $a_n = f(a_{n-1})$, is x such that $f(x) = x$.)

3. (30pts.) Consider the DDS:

$$a_n = 2a_{n-1} - 0.5(a_{n-1})^2$$

   (a) Show that $x = 0$ and $x = 2$ are equilibrium values of the DDS.

   (b) i. Using $a_0 = 0.05$, numerically generate the values of a_n for $n = 1, 2, \ldots, N$, say $N = 20$.
   (Use Mathematica to generate those numbers.)
   ii. Using $a_0 = -0.05$, numerically generate the values of a_n for $n = 1, 2, \ldots, N$.
   iii. Is the equilibrium value $x = 0$ *stable, unstable, neutral*, *semi-stable* (i.e. repels on one side of equilibrium value and attracts on the other side)? *Justify your answer*?

   (c) i. Using $a_0 = 1.5$, numerically generate the values of a_n for $n = 1, 2, \ldots, 20$.
   (Use Mathematica to generate those numbers.)
   ii. Using $a_0 = 2.8$, numerically generate the values of a_n for $n = 1, 2, \ldots, 20$.
   iii. Is the equilibrium value $x = 2$ *stable, unstable, neutral*, *semi-stable* (i.e. repels on one side of equilibrium value and attracts on the other side)? *Justify your answer*?

   (d) Sketch a cobweb diagram for this DDS.

4. (10pts.) Consider the DDS

$$a_n = 2a_{n-1} - 0.25(a_{n-1})^2 - 0.75$$

   The two equilibrium values are $x = 1$, which is unstable, and $x = 3$, which is stable. (You can double-check this if you wish.)

   Determine the maximum interval containing $x = 3$ such that if a_0 is on that interval, then

$$\lim_{n \to \infty} a_n = 3$$

   (Hint: Use the cobweb diagram as a tool.)

5. (20pts.) For obvious reasons, Farm-R-Us has been following the price of corn for several years. It turns out that when the price was \$8 per bushel in one year, the demand that year was 6000 bushels and the supply was 12,000 bushels the next year. Farm-R-Us also observed that when the price of corn was \$5 per bushel in one year, the demand that year was 9000 bushels and the supply was 10,000 bushels the following year.

   (a) From this information, determine the DDS for the supply of corn in any year n, knowing that

   $$s_n = r_s p_{n-1} + c$$

   where s_n is the number of bushels of corn supplied in year n and p_n is the price per bushel of corn in year n.
   (You should be able to find the constants r_s and c.)

   (b) From this information, determine the DDS for the demand of corn in any year n, knowing that

   $$d_n = -r_d p_n + b$$

   where d_n is the demand (in number of bushels) of corn in year n and p_n is the price per bushel of corn in year n.

   (c) Determine if an equilibrium price exists, and if so, determine its stability.

6. (**DDS version of population growth models.**) In sections 1.1 and 1.2, we considered exponential models for population growth; time, $t \geq 0$ was considered as a continuous variable.
   Now, let us considered time in a discrete way, i.e., $n = 0, 1, 2, 3, 4, 5 \ldots$

   (a) (20pts.) Suppose b and d are the birth and death rates, respectively, of a population, and suppose the change in population from the current time period to the next is the net growth rate times the current population. Let a_n be the population at time period n.
   The following equation is called the **difference equation** for this particular population model. The expression on the left side of the equation actually describes the *change* in the population between two consecutive time periods.

   $$a_n - a_{n-1} = (b - d)a_{n-1}$$

   The DDS for this particular population growth model is obtained by solving for a_n, i.e.

   $$a_n = (b - d + 1)a_{n-1}$$

   i. What type of DDS is this?
   ii. Solve the above DDS algebraically using the iterative method, assuming $b - d + 1$ is just a constant.
   iii. Suppose a_n describes the deer population in Stevens County n months from now, and suppose the net growth rate is $b - d = 0.2$ and right now we have $a_0 = 200$ deer.
   Use Mathematica to generate a table of a_n for $n = 1, 2, 3, \ldots, 20$.
   Does it look like the a_n is converging?
   iv. Under what circumstances (i.e what conditions on b, d) would deer become extinct in Stevens County?

(b) (20pts.) In the above model, we assumed that $b - d$ is a constant. Now, suppose $b - d$ changes as a_n changes and suppose we also have a sustainable capacity, m, for deer in Stevens County. (Somehow, deer have the instinct to flee to neighboring counties when they sense there is a lack of sustenance in Stevens County.)

Just as in Section 1.2, we assume that the rate of growth $b - d$ is proportional to

$$\left(1 - \frac{a_{n-1}}{m}\right) \quad \text{i.e} \quad b - d = k\left(1 - \frac{a_{n-1}}{m}\right)$$

Thus, when deer population is very low, the rate is almost constant, and when deer population is close to its capacity, the rate is close to 0. Hence, the **difference equation** for the population of deer is now:

$$a_n - a_{n-1} = k\left(1 - \frac{a_{n-1}}{m}\right) a_{n-1}$$

And, the DDS (a.k.a. the discrete logistics equation) is

$$a_n = k\left(1 - \frac{a_{n-1}}{m}\right) a_{n-1} + a_{n-1}$$

or

$$a_n = (k+1)a_{n-1} - \frac{k}{m}(a_{n-1})^2$$

(**FYI, the portion** $-\frac{k}{m}(a_{n-1})^2$ **is sometimes called the** damping term **since it prevents the population from growing without bound**).

i. What type of DDS is this?

ii. What are the equilibrium values? Are these what you would have expected?

iii. Suppose a_n describes the deer population (in units of thousand) in Stevens County n months from now, and suppose $k = 0.2$, $m = 8$ thousand, and right now we have $a_0 = 3000$ deer. Use Mathematica to generate a table of a_n for $n = 1, 2, 3, \ldots, N$.

iv. Plot a_n versus n.
Does it look like the a_n is converging? If so, to where and is that what you would expect?