MATH 3411 (Ng/Fall 2011)
Handout on the Ford-Fulkerson's Max-Flow Algorithm
November 22 & December 1, 2011.

Ford-Fulkerson's Maximum-Flow Algorithm

Given:

(i) A connected directed graph $G = (V, A)$, two distinguished vertices called *source* (s) and *sink* (t),

(ii) capacities $u_{ij} \geq 0$ on each arc $(i, j) \in A$.

Want: A maximum value $s \to t$ flow $(\bar{x}_{ij})$ that observes capacities, and such that net flows into all vertices except s and t are zero.

Initialize: Pick any feasible flow $\bar{x}$ (zero flow on each arc if no other is known).

Step 1 Define a residual graph, $\hat{G} = (V, \hat{A})$ and capacities $\hat{u}_{ij}$ with respect to the current flow, $\bar{x}$ in the following way:

$$\hat{A} = \hat{A}_1 \cup \hat{A}_2$$

where

$$\hat{A}_1 = \{(i, j) : (i, j) \in A, \bar{x}_{ij} < u_{ij}\}$$

i.e. all the arcs in original graph, whose current flow values are not at their capacities; and

$$\hat{A}_2 = \{(j, i) : (i, j) \in A, \bar{x}_{ij} > 0\}$$

i.e. create a new arc (j, i) if arc (i, j) in original graph has positive current flow.

If $(i, j) \in \hat{A}_1$ then assign **new capacity** $\hat{u}_{ij} = u_{ij} - \bar{x}_{ij}$

If $(j, i) \in \hat{A}_2$ then assign **new capacity** $\hat{u}_{ji} = \bar{x}_{ij}$

Step 2 Find an augmenting chain in G i.e. find an $s \to t$ directed path in $\hat{G}$.

If such directed path in $\hat{G}$ exists then go to Step 3; otherwise go to Step 4.

Step 3 Augmentation or update:

Compute $\delta \leftarrow min\{\hat{u}_{ij} : (i, j)$ is in the directed $s \to t$ path in $\hat{G}\}$.

Update

$$\bar{x}_{ij} \leftarrow \begin{cases} \bar{x}_{ij} + \delta & \text{if arc } (i,j) \text{ is forward on corresponding chain in } G \\ \bar{x}_{ij} - \delta & \text{if arc } (i,j) \text{ is reverse on corresponding chain in } G \\ \bar{x}_{ij} & \text{otherwise} \end{cases}$$

Repeat Step 1

Step 4 The current flow is optimal.

Define $S \leftarrow \{v \in V : v$ is reachable from s in $\hat{G}\}$. STOP.

Recall: A vertex u is *reachable from* a vertex w in a directed graph if there is a directed path from w to u **in** the same directed graph.