Ford-Fulkerson’s Maximum-Flow Algorithm

Given:
1. A connected directed graph $G = (V, A)$, two distinguished vertices called source $(s)$ and sink $(t)$,
2. capacities $u_{ij} \geq 0$ on each arc $(i, j) \in A$.

Want: A maximum value $s \rightarrow t$ flow $(\bar{x}_{ij})$ that observes capacities,
and such that net flows into all vertices except $s$ and $t$ are zero.

Initialize: Pick any feasible flow $\bar{x}$ (zero flow on each arc if no other is known).

Step 1 Define a residual graph $\hat{G} = (V, \hat{A})$ and capacities $\hat{u}_{ij}$ with respect to the current flow, $\bar{x}$ in the following way:

$$\hat{A} = \hat{A}_1 \cup \hat{A}_2$$

where

$$\hat{A}_1 = \{(i, j) : (i, j) \in A, \bar{x}_{ij} < u_{ij}\}$$

i.e. all the arcs in original graph, whose current flow values are not at their capacities; and

$$\hat{A}_2 = \{(j, i) : (i, j) \in A, \bar{x}_{ij} > 0\}$$

i.e. create a new arc $(j, i)$ if arc $(i, j)$ in original graph has positive current flow.

If $(i, j) \in \hat{A}_1$ then assign new capacity $\hat{u}_{ij} = u_{ij} - \bar{x}_{ij}$
If $(j, i) \in \hat{A}_2$ then assign new capacity $\hat{u}_{ji} = \bar{x}_{ij}$

Step 2 Find an augmenting chain in $\hat{G}$ i.e. find an $s \rightarrow t$ directed path in $\hat{G}$.
If such directed path in $\hat{G}$ exists then go to Step 3; otherwise go to Step 4.

Step 3 Augmentation or update:
Compute $\delta \leftarrow \min \left\{ \hat{u}_{ij} : (i, j) \text{ is in the directed } s \rightarrow t \text{ path in } \hat{G} \right\}$.
Update

$$\bar{x}_{ij} \leftarrow \begin{cases} \bar{x}_{ij} + \delta & \text{if arc } (i, j) \text{ is forward on corresponding chain in } G \\ \bar{x}_{ij} - \delta & \text{if arc } (i, j) \text{ is reverse on corresponding chain in } G \\ \bar{x}_{ij} & \text{otherwise} \end{cases}$$

Repeat Step 1

Step 4 The current flow is optimal.
Define $S \leftarrow \{v \in V : v \text{ is reachable from } s \text{ in } \hat{G}\}$. STOP.

Recall: A vertex $u$ is reachable from a vertex $w$ in a directed graph if there is a directed path from $w$ to $u$ in the same directed graph.