Ford-Fulkerson’s Maximum-Flow Algorithm

Given:
(i) A connected directed graph \( G = (V, A) \), two distinguished vertices called source \( s \) and sink \( t \),
(ii) capacities \( u_{ij} \geq 0 \) on each arc \( (i, j) \in A \).

Want: A maximum value \( s \to t \) flow \( (\bar{x}_{ij}) \) that observes capacities, and such that net flows into all vertices except \( s \) and \( t \) are zero.

Initialize: Pick any feasible flow \( \bar{x} \) (zero flow on each arc if no other is known).

Step 1 Define a residual graph \( \hat{G} = (V, \hat{A}) \) and capacities \( \hat{u}_{ij} \) with respect to the current flow, \( \bar{x} \) in the following way:

\[
\hat{A} = \hat{A}_1 \cup \hat{A}_2
\]

where
\[
\hat{A}_1 = \{(i, j) : (i, j) \in A, \bar{x}_{ij} < u_{ij}\}
\]
i.e. all the arcs in original graph, whose current flow values are not at their capacities; and
\[
\hat{A}_2 = \{(j, i) : (i, j) \in A, \bar{x}_{ij} > 0\}
\]
i.e. create a new arc \( (j, i) \) if arc \( (i, j) \) in original graph has positive current flow.

If \( (i, j) \in \hat{A}_1 \) then assign new capacity \( \hat{u}_{ij} = u_{ij} - \bar{x}_{ij} \)
If \( (j, i) \in \hat{A}_2 \) then assign new capacity \( \hat{u}_{ji} = \bar{x}_{ij} \)

Step 2 Find an augmenting chain in \( G \) i.e. find an \( s \to t \) directed path in \( \hat{G} \).

If such directed path in \( \hat{G} \) exists then go to Step 3; otherwise go to Step 4.

Step 3 Augmentation or update:
Compute \( \delta \leftarrow \min \{ \hat{u}_{ij} : (i, j) \text{ is in the directed } s \to t \text{ path in } \hat{G} \} \).
Update
\[
\bar{x}_{ij} \leftarrow \begin{cases} 
\bar{x}_{ij} + \delta & \text{if arc } (i, j) \text{ is forward on corresponding chain in } G \\
\bar{x}_{ij} - \delta & \text{if arc } (i, j) \text{ is reverse on corresponding chain in } G \\
\bar{x}_{ij} & \text{otherwise}
\end{cases}
\]
Repeat Step 1

Step 4 The current flow is optimal.
Define \( S \leftarrow \{ v \in V : v \text{ is reachable from } s \text{ in } \hat{G} \} \). STOP.

Recall: A vertex \( u \) is reachable from a vertex \( w \) in a directed graph if there is a directed path from \( w \) to \( u \) in the same directed graph.