Ford-Fulkerson’s Maximum-Flow Algorithm

Given:
(i) A connected directed graph $G = (V, A)$, two distinguished vertices called source ($s$) and sink ($t$),
(ii) capacities $u_{ij} \geq 0$ on each arc $(i, j) \in A$.

Want: A maximum value $s \rightarrow t$ flow $(\bar{x}_{ij})$ that observes capacities, and such that net flows into all vertices except $s$ and $t$ are zero.

Initialize: Pick any feasible flow $\bar{x}$ (zero flow on each arc if no other is known).

Step 1 Define a residual graph $\hat{G} = (V, \hat{A})$ and capacities $\hat{u}_{ij}$ with respect to the current flow, $\bar{x}$ in the following way:

$$\hat{A} = A_1 \cup A_2$$

where
$$A_1 = \{(i, j) : (i, j) \in A, \bar{x}_{ij} < u_{ij}\}$$
i.e. all the arcs in original graph, whose current flow values are not at their capacities; and

$$A_2 = \{(j, i) : (i, j) \in A, \bar{x}_{ij} > 0\}$$
i.e. create a new arc $(j, i)$ if arc $(i, j)$ in original graph has positive current flow.

If $(i, j) \in A_1$ then assign new capacity $\hat{u}_{ij} = u_{ij} - \bar{x}_{ij}$
If $(j, i) \in A_2$ then assign new capacity $\hat{u}_{ji} = \bar{x}_{ij}$

Step 2 Find an augmenting chain in $G$ i.e. find an $s \rightarrow t$ directed path in $\hat{G}$.
If such directed path in $\hat{G}$ exists then go to Step 3; otherwise go to Step 4.

Step 3 Augmentation or update:
Compute $\delta \leftarrow \min \{\hat{u}_{ij} : (i, j) \text{ is in the directed } s \rightarrow t \text{ path in } \hat{G}\}$. 
Update

$$\bar{x}_{ij} \leftarrow \begin{cases} 
\bar{x}_{ij} + \delta & \text{if arc } (i, j) \text{ is forward on corresponding chain in } G \\
\bar{x}_{ij} - \delta & \text{if arc } (i, j) \text{ is reverse on corresponding chain in } G \\
\bar{x}_{ij} & \text{otherwise}
\end{cases}$$

Repeat Step 1

Step 4 The current flow is optimal.
Define $S \leftarrow \{v \in V : v \text{ is reachable from } s \text{ in } \hat{G}\}$. STOP.

Recall: A vertex $u$ is reachable from a vertex $v$ in a directed graph if there is a directed path from $v$ to $u$ in the same directed graph.