MATH 3411 (Ng/Fall 2011)
Handout on Rationale for Shortest Walk/Path Algorithms
for class on **November 17, 2011.**

1. Dijkstra's Algorithm

Given a directed graph $G = (V, A)$, a *source* vertex $s \in V$, and costs $c_{ij} \geq 0$ for each $(i, j) \in A$, we want to find the shortest (directed) path from s to all vertices in G.

For every $i \in V$, the labels $\nu(i)$ in the Dijkstra's Algorithm (DA) can be **defined** and **interpreted** as being either

- lengths of the shortest paths from the source to vertex i, if vertex i is permanently labeled, or
- lengths of the shortest paths from the source to vertex i using only permanently labeled vertices, if vertex i is temporarily labeled.

(a) The algorithm's initialization step forces the above interpretation to apply at the beginning of the first main step.

(b) The main step's choice of the temporarily labeled vertex with least label value ensures the new permanent label will agree with the above interpretation.

(c) Choice of the temporarily labeled vertex with the least label also implies that if the new permanent vertex is part of the path defining the label of any remaining temporarily labeled vertex, it will be the last vertex in that path (other than the temporary vertex).

(d) The main step update of DA makes such temporary label conform to the above interpretation (including the effect of the new permanent vertex).

(e) Under the above interpretation, all labels will be the required lengths of the shortest (directed) paths in $O(|V|)$ major steps, i.e. in the worse case scenario, it needs a linear polynomial function of $|V|$ elementary operations.

(f) Effort per main step of the DA is at worst $O(|V|)$ steps. Thus total time complexity of DA is $O(|V|^2)$, because it consists of initial effort + iterative effort in main step $= O(|V|) + O(|V|)O(|V|)$.

2. Floyd-Warshall's Algorithm

Given a directed graph $G = (V, A)$ and real number costs c_{ij} for each $(i, j) \in A$, we want to find the shortest (directed) path from **all** vertices to **all** vertices in G.

As in DA, there is a **definition** and **interpretation** of the labels in Floyd-Warshall's A;gorithm (FWA).

For each $m = 1, 2, \ldots, |V|$, for every pair of vertices $i, j \in V$, $\nu^m(i, j)$ means the length of the shortest (directed) path from vertex i to vertex j using only vertices $k \leq m$ as intermediaries.

(a) The algorithm's initialization conforms to this interpretation with $m = 0$, i.e. lengths of shortest paths with no intermediary vertices.

(b) FWA's main update can be viewed as *keeping the least of the shortest path using only* $k \leq m-1$ as intermediates, versus the *walk formed by connecting the shortest* i *to* m *and* m *to* j *paths having* $k \leq m - 1$ *as intermediate vertices.*

(c) If the latter in the above is chosen, and it duplicates any vertex $\bar{k} \leq m - 1$ (and there are no negative weight circuits), the previous $(m - 1)st$ label $\nu^{m-1}(i, j)$ was not optimal. That is, the walk constructed is always a path, and the above interpretation applies.

(d) If a "diagonal" entry $\nu^m(i, i)$ is changed from 0 to a negative number for some i and during some iteration m, the graph has a negative weight circuit.

(e) FWA requires $O(|V|)$ major iterations (because the largest m is $|V|$), each of which requires $O(|V|^2)$ effort because there are two loops of $i = 1, 2, \ldots, |V|$ and $j = 1, 2, \ldots, |V|$.

Thus, total time complexity or effort for FWA is in $O(|V|^3)$, which is still a polynomial (efficent) time effort.

3. Miscellaneous Notes

(a) The existence of negative weight circuits means that the shortest directed path is not always the shortest directed walk.

(b) An *acyclic* graph is a directed graph with no directed cycles, thus it has no negative weight directed cycles.

(c) Longest path problems are much harder to solve in terms of time complexity because they imply shortest path problems with negative circuits.