MATH 3411 (Ng/Fall 2011)
Handout on Rationale for
Ford-Fulkerson Algorithm to solve Max $s-t$ Flow
for class on December 1, 2011.

The Ford-Fulkerson's maximum $s-t$ flow algorithm simultaneously computes a maximum $s-t$ flow and a minimum $s-t$ cut in a given connected directed graph $G=(V,A)$ with capacities $u_{ij} \geq 0$, $\forall\,(i,j) \in A$, and twp distinguish vertices s (source) and t (sink), where $s,t \in V$.

The algorithm centers around the notion of *augmenting $s-t$ chains* in G, consisting of *forward* (toward t) arcs at less than capacity and *reverse* (toward s) arcs at positive flows. Based on a feasible $s-t$ flow $\bar{x}$, the search for an augmenting $s-t$ chain in G is accomplished by constructing an *auxilliary graph*, $\hat{G}=(V,\hat{A})$ and capacities $\hat{u}_{ij}$ with respect to the current flow, $\bar{x}$ in the following way:

$$\hat{A} = \{(i,j) : (i,j) \in A, \bar{x}_{ij} < u_{ij}\} \cup \{(j,i) : (i,j) \in A, \bar{x}_{ij} > 0\}$$

The other key idea is is the *capacity* of an *$s-t$ cut set* which is the sum of the capacities in the forward (s to t) arcs of the cut set.

1. An initial $s-t$ flow $\bar{x}$ that is feasible and integer can always be found by assigning zero flows on every arc.

2. If a feasible $s-t$ flow $\bar{x}$ admits an augmenting $s-t$ chain in G, then this feasible $s-t$ flow CANNOT be maximal because it can be increased by at least

$$\delta \leftarrow \min\{\delta_1, \delta_2\}$$

$$\text{where} \qquad \delta_1 \leftarrow \min\{(\bar{u}_{ij} - \bar{x}_{ij}) : (i,j) \ \text{ is forward in the chain}$$
$$\delta_2 \leftarrow \min\{\bar{x}_{ij} : (i,j) \ \text{ is reverse in the chain}$$

   and $\delta > 0$ by the definition of $s-t$ augmenting chain.

   The reason being if we set

$$\tilde{x}_{ij} \leftarrow \begin{cases} \bar{x}_{ij} + \delta & \text{if arc } (i,j) \text{ is forward on corresponding chain in } G \\ \bar{x}_{ij} - \delta & \text{if arc } (i,j) \text{ is reverse on corresponding chain in } G \\ \bar{x}_{ij} & \text{otherwise} \end{cases}$$

   then $\tilde{x}$ is still a feasible $s-t$ flow and the value of this flow, $\nu(\tilde{x}) = \nu(\bar{x}) + \delta > \nu(\bar{x})$.

   N.B. Basically, what we have here is a **necessary** condition for a feasible $s-t$ flow $\bar{x}$ to be maximal. In other words, if a feasible $s-t$ flow $\bar{x}$ is maximal then it CANNOT admit any $s-t$ augmenting chain in G.

3. Every augmenting $s-t$ chain in the original graph G corresponds to an $s-t$ directed path in $\hat{G}$.

4. The value of any feasible $s-t$ flow $\bar{x}$, $\nu(\bar{x})$ which is defined as the net flow out of s or the net flow into t, can also be thought of as the sum of forward flows across any cut set, less the sum of reverse flows across the same cut set.

   Justification: Let $(S,\tilde{S})$ define some $s-t$ cut. In order for flow to get from s to t, it has to travel via arcs across S to $\tilde{S}$. Also, in order for flow to get from t to s, it has to travel via arcs across $\tilde{S}$ to S. Therefore, the value of this $s-t$ flow is

$$\nu(\bar{x}) = \sum_{\substack{(i,j):(i,j) \\ \text{is forward in cut}}} \bar{x}_{ij} - \sum_{\substack{(i,j):(i,j) \\ \text{is reverse in cut}}} \bar{x}_{ij}$$

5. Lemma (Weak Duality Lemma.) For **every** feasible $s-t$ flow $\bar{x}$ and for **every** $s-t$ cut,

$$\text{value of feasible } s-t \text{ flow, } \nu(\bar{x}) \leq \text{ capacity of } s-t \text{ cut}$$

   Sketch of Proof: In class.

6. Lemma (Strong Duality Lemma.)
   Let $\bar{x}$ be a feasible $s-t$ flow and let $(S, \tilde{S})$ define an $s-t$ cut.
   Suppose

$$\text{value of feasible } s-t \text{ flow } \nu(\bar{x}) = \text{ capacity of this } s-t \text{ cut}$$

   then the feasible flow is maximum and the cut is a minimum capacity $s-t$ cut.
   Proof: In class.

7. Theorem (Max-Flow Min-Cut Theorem) [Ford-Fulkerson [1957]]

$$\text{The value of a maximum } s-t \text{ flow } = \text{ capacity of the minimum } s-t \text{ cut}$$

8. When there is no augmenting chain in G, the set of vertices, say S, reachable from s in $\hat{G}$ does NOT include t, and thus induces an $s-t$ cut set. (The reason being that if t is reachable from s in $\hat{G}$ then there exists an augmenting chain in G.

9. The cut set that $(S, V \setminus S)$ induced when there are no augmenting chains in G has capacity equal to the current flow.

   N.B. Basically, the previous two comments give us a **sufficient** condition for a feasible $s-t$ flow $\bar{x}$ to be maximal. In other words, if there is no more $s-t$ augmenting chain in G for $\bar{x}$ to admit, then the feasible $s-t$ flow $\bar{x}$ is maximal.

10. When the Ford-Fulkerson algorithm stops, the $s-t$ feasible flow produced is maximum and the $s-t$ cut produced has minimum capacity.

11. If a feasible $s-t$ flow is integer, and all capacities u_{ij} are integer, then the maximum flow produced by F-F algorithm will be integer as well.

    The reason being that the only time flow value changes is when we change the value by $+\delta$ or $-\delta$. And δ is always integer if capacities are integer since

$$\delta = \min\{\delta_1, \delta_2\}$$

    where

$$\delta_1 \leftarrow \min\{(\bar{u}_{ij} - \bar{x}_{ij}) : (i, j) \text{ is forward in the chain}$$

$$\delta_2 \leftarrow \min\{\bar{x}_{ij} : (i, j) \text{ is reverse in the chain}$$