1. In each of the following problems, use induction to prove that each equation is true for every positive integer $n$.
   (a) $1 + 3 + 5 + \ldots + (2n + 1) = (n + 1)^2$
   (b) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$
   (c) $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \ldots + (-1)^{n+1}n^2 = (-1)^{n+1}\frac{n(n + 1)}{2}$
   (d) $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \ldots + \frac{1}{(2n - 1)(2n + 1)} = \frac{n}{2n + 1}$

2. Let $T_n$ be the following recursively defined sequence.
   
   $T_1 = 1, \ T_2 = 2; \ T_k = T_{k-1} + T_{k-2}$ for $k \geq 3$

   Show, by mathematical induction, that $T_n$, in a non-recursive form, can be written as:
   
   $T_n = \frac{1}{\sqrt{5}} \left\{ \left( \frac{1}{2} + \frac{1}{2} \sqrt{5} \right)^{n+1} - \left( \frac{1}{2} - \frac{1}{2} \sqrt{5} \right)^{n+1} \right\}$ for $n = 1, 2, 3, \ldots$

3. Use mathematical induction to verify each of the following inequalities.
   (a) $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ whenever $n$ is a positive integer greater than 1.
   (b) $\frac{1}{2n} \leq 1 \cdot 3 \cdot 5 \ldots (2n-1) \leq 2 \cdot 4 \cdot 6 \ldots (2n)$ for $n = 1, 2, \ldots$
   (c) $2^n \geq n^2$; for $n = 4, 5, 6, \ldots$
   (d) Suppose for any positive integer, $k$,
   
   $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$

   Show that $H_{2^n} \geq 1 + \frac{n}{2}$ for $n = 0, 1, 2, 3, \ldots$

   (Note: For those of you who have taken Calculus 2 or who are familiar with series, you may recognize $H_k$ as the partial sum of the Harmonic series. Having proved the aforementioned inequality, you can also provide an alternative proof of why the Harmonic series diverges.

4. By experimenting with small values of $n$, guess a formula for the given sum, i.e. write each expression as the sum of at most two terms or as one fraction. Then use induction to prove that your formula is unequivocally correct for any positive integer $n$.
   (a) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n + 1)}$
   (b) $4 \left( \frac{1}{2 \cdot 3} \right) + 8 \left( \frac{2}{3 \cdot 4} \right) + 16 \left( \frac{3}{4 \cdot 5} \right) + \ldots + 2^{n+1} \left[ \frac{n}{(n + 1)(n + 2)} \right]$