Extensions of Mangat’s randomized-response model

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Abstract

The randomized-response technique can be an effective survey method when collecting sensitive information. In this paper, we extend the model proposed by Mangat (J. Roy. Statist. Soc. Ser. B 56 (1994) 93) in two ways. First, we propose a Bayesian version of the model, which is applicable when prior information on $\pi$, the sensitive characteristic prevalence, is available. Our Bayesian approach can provide greatly-improved point estimators when compared to those obtained from maximum likelihood; furthermore, our approach provides estimators guaranteed to lie within the parameter space. Second, we extend Mangat’s procedure to include data obtained from a stratified-sampling protocol and show that both of our new stratified procedures—one non-Bayesian and one Bayesian—are more efficient than the one initially proposed by Mangat (1994) for a single population.

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1. Introduction

Questioning individuals about sexual orientation, criminal activity, abortion, or other sensitive topics, is a difficult task. Individuals are not often willing to liberally reveal such information in classical survey settings for fear of social stigma, yet, for public-health and
socioeconomic reasons, estimates of the prevalence of such behaviors are often needed. The randomized-response technique can be an effective survey method to find such estimates because individual anonymity is preserved. Initial randomized-response models proposed by Warner (1965) and Greenberg et al. (1969) presumed that two yes/no questions were provided for each respondent and that a randomization device was used to determine which question would be answered. Since the interviewer would not know the outcome of the device, participants would be encouraged to give truthful responses to a sensitive question. This technique has generated much interest in the statistical literature since the publication of Warner’s randomized-response model. We do not aim to provide a comprehensive literature review here. Instead, we refer the interested reader to the review-oriented references listed herein; namely, Chaudhuri and Mukerjee (1988) and Tracy and Mangat (1996).

A primary focus of this paper is the implementation of Bayesian methods using Mangat’s (1994) randomized-response procedure. This procedure is a simple variant of Warner’s that is often more efficient (Mangat, 1994; Bhargava and Singh, 2002). A serious deficiency, however, like many other randomized-response procedures, is that Mangat’s maximum likelihood estimator (MLE) for the sensitive characteristic prevalence, \( \pi \), can be negative when \( \pi \) is small. This is compounded by the problem that randomized-response techniques are often used when dealing with rare traits. Taking a Bayesian approach not only ameliorates this problem, but can provide far better estimates, especially when the number of respondents available is small. Our research complements the work of Winkler and Franklin (1979), Migon and Tachibana (1997), Spurrier and Padgett (1980), and Pitz (1980), who deal with applying standard Bayesian techniques to the models of Warner (1965) and Greenberg et al. (1969). In other Bayesian settings, O’Hagan (1987) derives Bayes linear estimators using a nonparametric approach, and Oh (1994) and Unnikrishnan and Kunte (1999) use MCMC methods. None of these authors address the Bayesian formulation of Mangat’s approach as we do here.

Another focus of our work is generalizing Mangat’s model to situations where data are collected according to a stratified-sampling protocol. We investigate two approaches: one non-Bayesian and one Bayesian. We prove the existence of a non-Bayesian stratified technique that is guaranteed to outperform Mangat’s procedure which treats the population as homogeneous. We then incorporate our Bayesian approach into this stratified-sampling extension to illustrate the benefits conferred when incorporating prior information.

In Section 2, we derive a Bayes estimator for \( \pi \), using Mangat’s approach and a beta prior distribution. Our posterior distribution for \( \pi \) and Bayes point estimator are expressed in closed form and can be computed easily with available software. In Section 3, we compare Mangat’s MLE and our Bayes estimator, on frequentist grounds, in terms of mean-squared error (i.e., risk). We show that the resulting Bayes estimator of \( \pi \) is often preferred to the MLE in realistic applications. In Section 4, we propose extensions of Mangat’s model to incorporate data from stratified sampling using non-Bayesian and Bayesian approaches. In Section 5, we conclude with a brief summary discussion.

2. Estimation

With a simple random sample of \( n \) respondents, Mangat (1994) proposes a method by which each individual is instructed to respond “yes” if he or she belongs to the sensitive
group of interest, say \( S \). If he or she does not belong to \( S \), the respondent is then required to use a Warner randomization device consisting of two different statements. One statement is “I belong to the sensitive group \( S \),” and the other statement is “I do not belong to the sensitive group \( S \).” In this stage, individuals are assigned to the two statements with probabilities \( p \) and \( 1 - p \), respectively, for \( 0 < p < 1 \), and he or she responds “yes” or “no” to the question represented by the outcome of the randomization device. The entire procedure is completed by the respondent unobserved by the interviewer. Since a “yes” response does not necessarily imply membership in \( S \), individuals are encouraged to provide truthful responses. We assume throughout that all responses are truthful.

2.1. Maximum likelihood estimation

We let \( Y_i = 1 \), if the \( i \)th respondent answers “yes;” \( Y_i = 0 \), otherwise, and assume that \( Y_1, Y_2, \ldots, Y_n \) are iid Bernoulli random variables with mean \( \lambda = p + 1 - p \). The likelihood function of \( \pi \), provided that \( \lambda \geq 1 - p \), is given by

\[
L(\pi | y_1, y_2, \ldots, y_n) = (p + 1 - p)^t [p(1 - \pi)]^{n - t},
\]

for \( y_i = 0, 1 \), where the sufficient statistic \( t = \sum_i y_i \) denotes the number of “yes” responses. Since \( T \sim \text{binomial}(n, \lambda) \), the unrestricted MLE of \( \pi \), by invariance, is

\[
\hat{\pi}_M = \frac{\hat{\lambda} - 1 + p}{p}, \tag{1}
\]

where \( \hat{\lambda} = T/n \). It is straightforward to show that \( \hat{\pi}_M \) is an unbiased estimator of \( \pi \) with variance

\[
V(\hat{\pi}_M) = \frac{\pi(1 - \pi)}{n} + \frac{(1 - \pi)(1 - p)}{np}. \tag{2}
\]

Mangat (1994) shows that his strategy is more efficient than Warner’s (1965) when \( p > \frac{1}{3} \). However, the estimator \( \hat{\pi}_M \) can be negative when \( \lambda < 1 - p \). In light of this, we define the restricted Mangat MLE to be

\[
\hat{\pi}^*_M = \frac{\hat{\lambda}^* - 1 + p}{p},
\]

where

\[
\hat{\lambda}^* = \begin{cases} 
1 - p, & \hat{\lambda} \leq 1 - p \\
\hat{\lambda}, & 1 - p < \hat{\lambda} \leq 1.
\end{cases}
\]

Essentially, our restricted estimator \( \hat{\pi}^*_M \) is the minimum-distance projection of \( \hat{\pi}_M \) onto the unit interval, i.e. \( \hat{\pi}^*_M = 0 \), when \( \hat{\lambda} \leq 1 - p \) and \( \hat{\pi}^*_M = \hat{\pi}_M \), when \( \hat{\lambda} > 1 - p \). Clearly, the restricted estimator \( \hat{\pi}^*_M \) is no longer unbiased for \( \pi \). Furthermore, results presented by Moors (1981) can be used to argue that the restricted estimator \( \hat{\pi}^*_M \) is not admissible since the range of \( \hat{\pi}^*_M \) is not a subspace of \((0, 1)\).
2.2. Bayes estimation

We now formulate a Bayesian approach to estimate \( \pi \) using Mangat’s procedure. In many instances, the researcher has prior knowledge about the value \( \pi \) before data are collected. For example, using 1996 data, the percentage of women in the United States aged 15–44 having legal abortions is known to vary between 0.2 and 3.9 percent among different states (United States Centers for Disease Control and Prevention: Abortion Surveillance—United States, 1996, July 30, 1999), and the prevalence of homosexuality among American males is reported to be somewhere between 1 and 10 percent (Kinsey Institute; located online at http://www.kinseyinstitute.org/). Clearly, in these and other situations, prior information on \( \pi \) is available and should be incorporated into the estimation procedure.

Specifically, we assume that \( \pi \) is best regarded as a random variable that varies according to a beta distribution given by

\[
f_{\Pi}(\pi|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \pi^{\alpha-1}(1-\pi)^{\beta-1} I(0 < \pi < 1),
\]

where the parameters \( \alpha \) and \( \beta \) are constants larger than zero. For example, in the case of rare traits, such as abortion or homosexuality, the researcher most likely would prefer to have a relatively low risk in a subinterval closer to zero. This may be accomplished by selecting a prior distribution with \( \alpha < \beta \) and \( \pi \approx \alpha/\alpha + \beta \); this places more weight on small values of \( \pi \) and less weight on larger values of \( \pi \) (see also Lee, 1989). Of course, specification of model hyperparameters can be subjective. Appropriate model-checking analyses should always be performed as to assess the sensitivity of the prior model. For a further discussion on these issues, the reader is referred to Gelman et al., 2003.

Conditional on \( \Pi = \pi \), the number of “yes” responses in the Mangat procedure, \( T \), follows a binomial distribution with parameters \( n \) and \( \lambda = p\pi + 1 - p \). We write this distribution as

\[
f_{T|\Pi}(t|\pi; n, p) = \binom{n}{t} (p\pi + 1 - p)^t [p(1 - \pi)]^{n-t} = \binom{n}{t} p^n (1 - \pi)^{n-t} (\pi + d)^t = \binom{n}{t} p^n (1 - \pi)^{n-t} \sum_{j=0}^{t} \binom{t}{j} d^{t-j} \pi^j,
\]

for \( t = 0, 1, \ldots, n \), where \( d = (1 - p)/p \). The last step follows from writing the term \( (\pi + d)^t \) in its binomial expansion. The joint distribution of \( T \) and \( \Pi \) is thus given by

\[
f_{T,\Pi}(t, \pi|\alpha, \beta, n, p) = f_{\Pi}(\pi|\alpha, \beta) \times f_{T|\Pi}(t|\pi; n, p)
\]

\[
= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \pi^{\alpha-1}(1-\pi)^{\beta-1} \times \binom{n}{t} p^n (1 - \pi)^{n-t} \sum_{j=0}^{t} \binom{t}{j} d^{t-j} \pi^j.
\]

\[
= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \binom{n}{t} p^n (1 - \pi)^{n+\beta-t-1} \sum_{j=0}^{t} \binom{t}{j} d^{t-j} \pi^{\alpha+j-1},
\]
for \( t = 0, 1, \ldots, n \) and \( 0 < \pi < 1 \), and the marginal distribution of \( T \), obtained by integrating out over \( \pi \), is
\[
     f_T(t|\alpha, \beta, n, p) = \int_0^1 f_{T, \Pi}(t, \pi|\alpha, \beta, n, p) \, d\pi
\]
\[
     = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{n}{t} \right) p^n \sum_{j=0}^{t} \binom{t}{j} \pi^j (1 - \pi)^{n-j-t} \int_0^1 \pi^x \, d\pi
\]
\[
     = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{n}{t} \right) p^n \sum_{j=0}^{t} \binom{t}{j} d^{n-j} B(\alpha + j, n + \beta - t)
\]
for \( t = 0, 1, \ldots, n \), where \( B(\alpha + j, n + \beta - t) = \frac{\Gamma(\alpha + j)\Gamma(n + \beta - t)}{\Gamma(n + \alpha + \beta + j - t)} \) and \( \Gamma(\cdot) \) is the usual gamma function. The posterior distribution is available in closed-form; standard calculations show that
\[
     f_{\Pi|T}(\pi|t; \alpha, \beta, n, p) = \frac{f_{T, \Pi}(t, \pi|\alpha, \beta, n, p)}{f_T(t|\alpha, \beta, n, p)}
\]
\[
     = \frac{\sum_{j=0}^{t} \binom{t}{j} d^{n-j} \pi^x \pi^{n-j-t} B(\alpha + j, n + \beta - t)}{\sum_{j=0}^{t} \binom{t}{j} d^{n-j} B(\alpha + j, n + \beta - t)} I(0 < \pi < 1).
\]

With the posterior distribution \( f_{\Pi|T}(\pi|t; \alpha, \beta, n, p) \) and a given loss function, say, \( L(\pi, a) \), the Bayes estimator of \( \pi \) with respect to \( L(\pi, a) \) is the value of \( a \) that minimizes
\[
     E[L(\Pi, a)|T = t, \alpha, \beta] = \int_0^1 L(\pi, a) f_{\Pi|T}(\pi|t; \alpha, \beta, n, p) \, d\pi.
\]

Using \( L(\pi, a) = (\pi - a)^2 \), i.e. squared-error loss, the Bayes estimator of \( \pi \) is given by the mean of posterior distribution \( f_{\Pi|T}(\pi|t; \alpha, \beta, n, p) \). For the remainder of this section, and for all comparisons in Sections 3 and 4, we shall assume the use of this particular loss function. We discuss the use of other loss functions briefly in Section 5. A closed-form expression for the Bayes estimator, under squared-error loss, is given by
\[
     \hat{\pi}_B = \int_0^1 \pi f_{\Pi|T}(\pi|t; \alpha, \beta, n, p) \, d\pi
\]
\[
     = \frac{\sum_{j=0}^{t} \binom{t}{j} d^{n-j} B(\alpha + j + 1, n + \beta - t)}{\sum_{j=0}^{t} \binom{t}{j} d^{n-j} B(\alpha + j, n + \beta - t)}.
\]

Our Bayesian estimator \( \hat{\pi}_B \) is much more computationally-involved than either of the maximum likelihood estimators. To ease this burden, we have written a simple program in MATLAB; this program is available from the first author. However, a desirable trait associated with our Bayes estimator, unlike the maximum likelihood estimators, is that its value
will always fall within the parameter space, even at the extreme realization of \( t = 0 \), in which case \( \hat{\pi}_B = \alpha / (\alpha + n + \beta) \). For this value of \( t \), \( \hat{\pi}_M < 0 \) and \( \hat{\pi}_M^* = 0 \), which are largely nonsensical estimates.

Before we compare the Bayesian and non-Bayesian approaches, we make a brief remark about the use of noninformative priors. Of course, in lieu of any prior information about \( \pi \), the researcher can take \( \alpha = \beta = 1 \). Alternatively, one could define a noninformative prior using Jeffreys’ principle by specifying \( f_{II}(\pi) \propto [p^2(p\pi + 1 - p)^{-1} + p(1 - \pi)^{-1}]^{1/2} \); however, in this instance, the estimator \( \hat{\pi}_B \) no longer exists in closed form. For the remainder of the manuscript, we will assume the prior distribution in (3).

\[ \text{3. Point estimate comparisons} \]

In order to assess the impact of including prior information, we compare the Bayes estimator \( \hat{\pi}_B \) to both the unrestricted and restricted Mangat maximum likelihood estimators, \( \hat{\pi}_M \) and \( \hat{\pi}_M^* \), respectively. Of course, we are considering estimators derived using different approaches; thus, it is important to be cognizant of exactly how the comparison is made. Samaneigo and Reneau (1994) present an interesting approach of comparing Bayesian and frequentist point estimators by making use of Bayes risk relative to an unknown “true” prior distribution. Unfortunately, their approach is not applicable here since \( \hat{\pi}_B \) cannot be expressed as a linear function of \( \hat{\pi}_M \) or \( \hat{\pi}_M^* \) (see Theorem 1 in Samaneigo and Reneau, 1994). Thus, we make the comparison on frequentist terms and do not consider the loss function in the comparison, as in Chaubey and Li (1995). For a fixed \( \pi \), the mean-squared errors of \( \hat{\pi}_B \) and \( \hat{\pi}_M \) are given by

\[
\text{MSE}(\hat{\pi}_B) = ET[II \left( \hat{\pi}_B - \pi \right)^2] = \sum_{t=0}^{n} (\hat{\pi}_B - \pi)^2 \times \left( \begin{array}{c} n \\ t \end{array} \right) (p\pi + 1 - p)^t [p(1 - \pi)]^{n-t},
\]

and

\[
\text{MSE}(\hat{\pi}_M) = \sum_{t=0}^{n} (\hat{\pi}_M - \pi)^2 \times \left( \begin{array}{c} n \\ t \end{array} \right) (p\pi + 1 - p)^t [p(1 - \pi)]^{n-t}, \tag{5}
\]

respectively. Similarly, \( \text{MSE}(\hat{\pi}_M^*) \) is computed as in Eq. (5), with \( \hat{\pi}_M^* \) replacing \( \hat{\pi}_M \). The reader will recall that since \( \hat{\pi}_M \) is unbiased, \( \text{MSE}(\hat{\pi}_M) \) is also given by (2).

Fig. 1 displays the values of \( \text{MSE}(\hat{\pi}_B) \), \( \text{MSE}(\hat{\pi}_M) \), and \( \text{MSE}(\hat{\pi}_M^*) \), for the case wherein \( n = 25 \), \( p = 0.6 \) or \( p = 0.8 \), and \( \pi \) ranging from 0 to 0.20. We have chosen four prior distributions, three of which have mean \( \alpha / (\alpha + \beta) = 0.05: (\alpha, \beta) = (0.5, 9.5), (1, 19) \), and \( (2, 38) \). We have also included the \((\alpha, \beta) = (1, 1)\) case to examine the performance of the model using a noninformative prior distribution. When examining the informative priors, for either the \( p = 0.6 \) or the \( p = 0.8 \) case, one will note the overall large reduction in mean-squared error (MSE) for the Bayes estimator when compared to the unrestricted and restricted MLE, especially when \( \pi \leq 0.10 \). As expected, when \( \pi = 0.05 \) the reduction is greatest. When \( p = 0.6 \), the Bayes estimators continue to outperform their MLE competitors...
even when $\pi$ is larger than 0.10. When $p = 0.8$, the MLE and restricted MLE do outperform the Bayes estimators when $\pi$ gets close to 0.2. This notwithstanding, Fig. 1 shows that, to a large degree, the Bayes estimators in this setting are largely robust when the prior is not misspecified too greatly. We have observed similar results when using other less informative priors as well. In fact, even for the $(\alpha, \beta) = (1, 1)$ case, we see that the Bayes estimator always outperforms the unrestricted MLE when $p = 0.6$.

Fig. 2 displays mean-squared errors of the maximum likelihood and Bayes estimators in the same setting as Fig. 1, with $p$ fixed at 0.6, for $n = 100$ and 250. These are sample sizes that may be more realistic in practice (although it is not uncommon for $n$ to be small when dealing with sensitive issues). For the informative priors, when $n = 100$, the Bayes estimators outperform both maximum likelihood estimators for $\pi \leq 0.14$, even when priors are chosen to have mean $\pi = 0.05$. When $n = 250$, the reduction in MSE realized by using a Bayes procedure diminishes noticeably; however, all estimators based on informative priors still continue to have smaller MSE when $0.01 < \pi < 0.10$. In light of LeCam’s (1958) results concerning the convergence of posterior distributions to a normal distribution, one would expect that for larger values of $n$, the reduction in MSE realized by the Bayes procedure
would be even smaller than those when $n=250$ (Spurrier and Padgett, 1980). This follows since both $\hat{\pi}_M$ and $\hat{\pi}_M^*$ will be approximately normal, provided that $n$ is large and $\pi$ is bounded away from zero. Finally, as one would expect, the estimators $\hat{\pi}_M$ and $\hat{\pi}_M^*$ behave more similarly for larger values of $\pi$.

Figs. 1 and 2 were constructed using the MatLab software package. A copy of the program used to create these figures is available on the first author’s website located at http://mrs.umn.edu/~jongmink/. Researchers wanting to experiment with different values of $n$, $p$, $x$, $\beta$, and $\pi$ can download and run these programs to determine whether or not a Bayesian approach would be preferred. That the advantages of a Bayesian approach diminishes as $n$ gets larger has motivated the authors to consider extending Mangat’s randomized-response model to incorporate data from stratified sampling. This is the subject of Section 4.

4. Two stratified models

The use of stratification can greatly strengthen the sampling protocol. Not only can stratification decrease the variability of an overall estimator, but it may be preferred when
interest lies in considering individuals in strata separately. For example, an investigator may wish to ascertain information about the prevalence of homosexuality among individuals of different races or income classes, as seen in Diamond (1993). Or, in a public-health study involving abortion, the researcher may wish to compare abortion rates among women of different educational levels, age groups, or some other factor. In this section, we extend Mangat’s randomized-response model to incorporate data in these types of settings.

We assume that the population of interest is divided up into \( k \geq 1 \) strata and that a simple random sample of individuals of size \( n_i \) is taken from stratum \( i \). We assume that sample sizes \( n_i \) are fixed a priori and denote the total sample size by \( n = \sum_i n_i \). In addition, we denote the size of stratum \( i \) by \( N_i \), the population size by \( N = \sum_i N_i \), and assume that \( N_1, N_2, \ldots, N_k \) are known. Our stratified approach presumes that individuals selected in stratum \( i \) follow the Mangat procedure, within stratum \( i \) and independently of other strata, according to a randomization device. The randomization probability in the procedure for stratum \( i \) is denoted as \( p_i \), \( 0 < p_i < 1 \), to emphasize that such probabilities may be different for different strata. We denote the prevalence of the sensitive characteristic in stratum \( i \) by \( \pi_i \) and the overall prevalence by \( \pi \).

### 4.1. Non-Bayesian approach

Our approach here is to implement an optimal allocation of individuals across strata, similar to the approach taken by Kim and Warde (2004) in the case of Warner’s (1965) model. For further information on optimal allocation, we refer the reader to Kim and Warde (2004). Under our stratified Mangat approach, the probability of a “yes” response in stratum \( i \) is given by \( \lambda_i = p_i \pi_i + 1 - p_i \), and the unrestricted MLE of \( \pi_i \) is

\[
\hat{\pi}_S = \frac{\hat{\lambda}_i - 1 + p_i}{p_i},
\]

where \( \hat{\lambda}_i = T_i / n_i \) is the proportion of “yes” responses obtained from the \( i \)th sample. It follows that \( \hat{\pi}_S \) is an unbiased estimator for \( \pi_i \); furthermore, with \( w_i = N_i / N \), for \( i = 1, 2, \ldots, k \), and unbiased estimator of \( \pi = \sum_i w_i \pi_i \) is given by

\[
\hat{\pi}_S = \sum_{i=1}^k w_i \hat{\pi}_S = \sum_{i=1}^k w_i \left( \frac{\hat{\lambda}_i - 1 + p_i}{p_i} \right).
\]

Under our independent assumption among strata, it follows from (2) that the variance of \( \hat{\pi}_S \) is given by

\[
V(\hat{\pi}_S) = \sum_{i=1}^k w_i^2 V(\hat{\pi}_S) = \sum_{i=1}^k w_i^2 \left[ \frac{\pi_i(1 - \pi_i)}{n_i} + \frac{(1 - \pi_i)(1 - p_i)}{n_i p_i} \right].
\]
Using the optimal-allocation approach of Kim and Warde (2004), one can show that the variance in (7) is minimized when \( n_1, n_2, \ldots, n_k \) are chosen such that

\[
\frac{n_i}{n} = \frac{w_i \left[ \pi_i (1 - \pi_i) + \frac{(1 - \pi_i)(1 - p_i)}{p_i} \right]^{1/2}}{\sum_i w_i \left[ \pi_i (1 - \pi_i) + \frac{(1 - \pi_i)(1 - p_i)}{p_i} \right]^{1/2}}.
\]

Under this optimal-allocation assumption, straightforward algebra shows that the variance in (7) becomes

\[
V(\hat{\pi}_S) = \frac{1}{n} \left[ \sum_{i=1}^k w_i \sqrt{\pi_i (1 - \pi_i) + \frac{(1 - \pi_i)(1 - p_i)}{p_i}} \right]^2.
\]  

(8)

Of course, using the optimal-allocation strategy requires one to provide reliable estimates of \( \pi_1, \pi_2, \ldots, \pi_k \). However, in many settings, such as those in the beginning of Section 2, these estimates are often available from preliminary data or other historical information.

**Theorem 4.1.** Assuming optimal allocation, the stratified estimator \( \hat{\pi}_S \) is more efficient than the Mangat estimator \( \hat{\pi}_M \) when \( p = p_1 = p_2 = \cdots = p_k \neq 0 \).

**Proof.** For brevity, we prove the result for the \( k=2 \) case. When \( k > 2 \), the proof is analogous. Suppose that \( p = p_1 = p_2 \neq 0 \), \( n = n_1 + n_2 \), \( \hat{\pi}_S = w_1\hat{\pi}_{S_1} + w_2\hat{\pi}_{S_2} \), where \( w_i = N_i/N \) for \( i = 1, 2 \), and that optimal allocation is used so that \( V(\hat{\pi}_S) \) is given by (8). It suffices to show that \( V(\hat{\pi}_M) - V(\hat{\pi}_S) > 0 \). We have

\[
V(\hat{\pi}_M) - V(\hat{\pi}_S) = \frac{\pi(1 - \pi)}{n} + \frac{(1 - \pi)(1 - p)}{np} - \frac{1}{n} \left[ \sum_{i=1}^2 w_i \sqrt{\pi_i (1 - \pi_i) + \frac{(1 - \pi_i)(1 - p_i)}{p_i}} \right]^2.
\]  

(9)

Inserting \( \pi = w_1\pi_1 + w_2\pi_2 \) into (9), we obtain

\[
V(\hat{\pi}_M) - V(\hat{\pi}_S) = \frac{w_1w_2}{n} \left[ \left( \sqrt{\pi_1 (1 - \pi_1) + \frac{(1 - \pi_1)(1 - p_1)}{p_1}} - \sqrt{\pi_2 (1 - \pi_2) + \frac{(1 - \pi_2)(1 - p_2)}{p_2}} \right)^2 + (\pi_1 - \pi_2)^2 \right] \geq 0
\]

with equality holding when \( \pi_1 = \pi_2 = \pi \). Thus, the result follows. \( \square \)

Theorem 4.1 guarantees that when optimal-allocation is used, the stratified estimator in (6) is more efficient than Mangat’s estimator in (1) which ignores the stratification (it should be noted that when optimal allocation is not used, this is not necessarily true). Thus, in light of Theorem 4.1, we consider it of interest to examine how great the benefits of stratification actually are. Table 1 displays values of \( \text{RE}(\hat{\pi}_M) \) to \( \hat{\pi}_S \) = \( V(\hat{\pi}_M)/V(\hat{\pi}_S) \), the
Table 1
Relative efficiency calculations, $\text{RE}(\hat{\pi}_M \text{ to } \hat{\pi}_S)$, for $\pi = 0.10$, $w_i$ constant across strata, and different values of $p$ when using optimal allocation

<table>
<thead>
<tr>
<th>Individual stratum proportions</th>
<th>$p = 0.6$</th>
<th>$p = 0.7$</th>
<th>$p = 0.8$</th>
<th>$p = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\pi_1, \pi_2) = (0.08, 0.12)$</td>
<td>1.0006</td>
<td>1.0009</td>
<td>1.0016</td>
<td>1.0034</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2) = (0.05, 0.15)$</td>
<td>1.0037</td>
<td>1.0057</td>
<td>1.0100</td>
<td>1.0220</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2) = (0.02, 0.18)$</td>
<td>1.0094</td>
<td>1.0147</td>
<td>1.0260</td>
<td>1.0593</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3) = (0.08, 0.10, 0.12)$</td>
<td>1.0004</td>
<td>1.0006</td>
<td>1.0011</td>
<td>1.0023</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3) = (0.05, 0.10, 0.15)$</td>
<td>1.0024</td>
<td>1.0038</td>
<td>1.0066</td>
<td>1.0146</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3) = (0.02, 0.10, 0.18)$</td>
<td>1.0063</td>
<td>1.0097</td>
<td>1.0172</td>
<td>1.0390</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3, \pi_4) = (0.07, 0.09, 0.11, 0.13)$</td>
<td>1.0008</td>
<td>1.0013</td>
<td>1.0023</td>
<td>1.0050</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3, \pi_4) = (0.04, 0.08, 0.12, 0.16)$</td>
<td>1.0029</td>
<td>1.0045</td>
<td>1.0080</td>
<td>1.0176</td>
</tr>
</tbody>
</table>

relative efficiency of using Mangat’s estimator in (1), which assumes no stratification, to the stratified estimator in (6). Note that $\text{RE}(\hat{\pi}_M \text{ to } \hat{\pi}_S)$ does not depend on $n$. In all cases in Table 1, we have assumed that optimal allocation is used and that the randomization probability $p_i$ and the stratum weights $w_i$ are constant across all strata. From Theorem 4.1, we know that $\text{RE}(\hat{\pi}_M \text{ to } \hat{\pi}_S)$ will always be larger than one. However, the reader will note that the benefits of using the stratified estimator in (6) are simply not that great. For example, when $\pi = 0.10$, $(\pi_1, \pi_2, \pi_3) = (0.05, 0.10, 0.15)$, $p = 0.6$, and $w_1 = w_2 = w_3 = \frac{1}{3}$, $\hat{\pi}_S$ is only marginally more efficient than $\hat{\pi}_M$ as $\text{RE}(\hat{\pi}_M \text{ to } \hat{\pi}_S) = 1.0024$. Thus, while our stratified estimator $\hat{\pi}_S$ is guaranteed to be more efficient when using optimal allocation, the results from Table 1 enervate this non-Bayesian approach. The use of $\hat{\pi}_S$ does not confer substantial gains in efficiency that one might expect when compared to the performance of the estimator in (1). In fact, from a practical standpoint, the small gain in efficiency may not be worth the added complexity of using a stratified-sampling design.

4.2. Bayesian approach

In light of the results from Section 4.1, our goal is to incorporate the Bayesian approach of Section 2.2 into a stratified-sampling situation. We know of no other work in the randomized-response literature that considers a stratified Bayesian approach as we do here. However, we consider it of interest, as in the single-population case, to explore the potential benefits of incorporating prior information. Assuming a beta($\alpha_i$, $\beta_i$) prior distribution for $\pi_i$ in stratum $i$, with $w_i = N_i / N$ and $n = \sum_i n_i$, our stratified Bayes estimator is given by

$$\hat{\pi}_{SB} = \sum_{i=1}^{k} w_i \hat{\pi}_{B_i},$$

where $\hat{\pi}_{B_i}$ is the Bayes estimator in (4) using individuals only from stratum $i$. For the remainder of this manuscript, we will assume that optimal allocation of the individuals is
used. The MSE of our stratified Bayesian estimator is given by

\[
\text{MSE}(\hat{\pi}_{SB}) = \sum_{i=1}^{k} \sum_{t_i=0}^{n_i} w_i^2 (\hat{\pi}_{SB} - \pi_i)^2 \left( \frac{n_i}{t_i} \right) (p_i \pi_i + 1 - p_i)^{t_i} [p_i (1 - \pi_i)]^{n_i - t_i}
\]

\[
+ \left[ \sum_{i=1}^{k} \sum_{t_i=0}^{n_i} w_i (\hat{\pi}_{SB} - \pi_i) \left( \frac{n_i}{t_i} \right) (p_i \pi_i + 1 - p_i)^{t_i} [p_i (1 - \pi_i)]^{n_i - t_i} \right]^2,
\]

where \( t_i \) is number of “yes” responses in stratum \( i \). As in Section 4.1, we compare our stratified estimator to Mangat’s estimator \( \hat{\pi}_M \). We define the relative efficiency measure

\[
\text{RE}(\hat{\pi}_M, \hat{\pi}_{SB}) = \frac{\text{MSE}(\hat{\pi}_M)}{\text{MSE}(\hat{\pi}_{SB})}.
\]

Again, one will note that since \( \hat{\pi}_M \) is unbiased, \( \text{MSE}(\hat{\pi}_M) \) is also given by (2).

Tables 3 and 4 contain values of \( \text{RE}(\hat{\pi}_M, \hat{\pi}_{SB}) \) for \( n = 150 \) and 300, respectively. In each situation, we have chosen prior distributions according to those in Table 2; prior distributions were chosen so that \( \pi_i = \alpha_i / (\alpha_i + \beta_i) \). For example, when \( \pi_i = 0.10 \), we have chosen \( (\alpha_i, \beta_i) = (2, 18) \); this case is not listed in Table 2. In addition, we have assumed equal stratum weights \( w_i \) and a constant randomization probability \( p_i = p \) for all strata, as in Table 1. As one can see from Tables 3 and 4, the potential gains from using a stratified Bayesian estimator are enormous, especially for smaller \( n \) and smaller \( p \). For example, when \( \pi = 0.10, n = 150, \ (\alpha_1, \alpha_2, \alpha_3) = (0.05, 0.10, 0.15), \ p = 0.6, \) and \( w_1 = w_2 = w_3 = \frac{1}{3}, \hat{\pi}_{SB} \) is 18.88 times more efficient than \( \hat{\pi}_M \) when stratum prior distributions are chosen as in Table 2. Recall that this figure was only marginally larger than unity with the non-Bayesian approach. The increased efficiency that results from incorporating prior information extends to the situation wherein data are collected according to this stratified protocol.

5. Discussion

In this paper, we have extended the Mangat (1994) approach so that one can include prior information about a sensitive characteristic parameter \( \pi \) and have illustrated the benefits conferred from doing so. In addition, we have proposed two stratified extensions of Mangat’s model. Our findings are that a stratified Bayesian estimator is largely preferred to its non-Bayesian analogue in realistic settings.
Table 4
Relative efficiency calculations, RE($\hat{p}_M$ to $\hat{p}_{SB}$), for $\pi = 0.10, n = 150, w_i$ constant across strata, and different values of $p$ when using optimal allocation

<table>
<thead>
<tr>
<th>Individual stratum proportions</th>
<th>$p = 0.6$</th>
<th>$p = 0.7$</th>
<th>$p = 0.8$</th>
<th>$p = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\pi_1, \pi_2) = (0.08, 0.12)$</td>
<td>13.34</td>
<td>7.88</td>
<td>4.57</td>
<td>2.43</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2) = (0.05, 0.15)$</td>
<td>11.66</td>
<td>7.28</td>
<td>4.48</td>
<td>2.57</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2) = (0.02, 0.18)$</td>
<td>33.86</td>
<td>19.36</td>
<td>10.78</td>
<td>5.47</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3) = (0.08, 0.10, 0.12)$</td>
<td>21.36</td>
<td>11.92</td>
<td>6.41</td>
<td>3.02</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3) = (0.05, 0.10, 0.15)$</td>
<td>18.88</td>
<td>10.92</td>
<td>6.10</td>
<td>3.02</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3) = (0.02, 0.10, 0.18)$</td>
<td>34.03</td>
<td>18.88</td>
<td>10.11</td>
<td>4.72</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3, \pi_4) = (0.07, 0.09, 0.11, 0.13)$</td>
<td>155.55</td>
<td>78.11</td>
<td>36.73</td>
<td>14.13</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3, \pi_4) = (0.04, 0.08, 0.12, 0.16)$</td>
<td>38.28</td>
<td>20.70</td>
<td>10.63</td>
<td>4.63</td>
</tr>
</tbody>
</table>

Table 3
Relative efficiency calculations, RE($\hat{p}_M$ to $\hat{p}_{SB}$), for $\pi = 0.10, n = 300, w_i$ constant across strata, and different values of $p$ when using optimal allocation

<table>
<thead>
<tr>
<th>Individual stratum proportions</th>
<th>$p = 0.6$</th>
<th>$p = 0.7$</th>
<th>$p = 0.8$</th>
<th>$p = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\pi_1, \pi_2) = (0.08, 0.12)$</td>
<td>5.55</td>
<td>3.61</td>
<td>2.34</td>
<td>1.46</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2) = (0.05, 0.15)$</td>
<td>5.40</td>
<td>3.71</td>
<td>2.55</td>
<td>1.66</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2) = (0.02, 0.18)$</td>
<td>13.07</td>
<td>8.25</td>
<td>5.19</td>
<td>3.06</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3) = (0.08, 0.10, 0.12)$</td>
<td>8.13</td>
<td>4.97</td>
<td>2.97</td>
<td>1.63</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3) = (0.05, 0.10, 0.15)$</td>
<td>7.69</td>
<td>4.88</td>
<td>3.03</td>
<td>1.73</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3) = (0.02, 0.10, 0.18)$</td>
<td>12.80</td>
<td>7.80</td>
<td>4.64</td>
<td>2.51</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3, \pi_4) = (0.07, 0.09, 0.11, 0.13)$</td>
<td>48.75</td>
<td>26.32</td>
<td>13.46</td>
<td>5.76</td>
</tr>
<tr>
<td>$(\pi_1, \pi_2, \pi_3, \pi_4) = (0.04, 0.08, 0.12, 0.16)$</td>
<td>13.75</td>
<td>8.10</td>
<td>4.62</td>
<td>2.33</td>
</tr>
</tbody>
</table>

Our entire discussion has presumed a squared-error loss function, but other loss functions could be used. In the single-population case, we have found that the shape of the posterior distribution, $f_{\Pi|T}(\pi|t; \alpha, \beta, n, p)$, is generally unimodal and often highly skewed right. In light of this, one might prefer the mean $\hat{\pi}_B$ over, say, the median or mode of $f_{\Pi|T}(\pi|t; \alpha, \beta, n, p)$ if underestimation is more severe, as it often is in public-health studies. Alternatively, one could penalize underestimation by using a linear loss function of the form $L(\pi, a) = \gamma_0(\pi - a)I(\pi \geq a) + \gamma_1(a - \pi)I(\pi < a)$, for suitably chosen values of $\gamma_0 > \gamma_1$. In this situation, the resulting Bayes estimate is the $\gamma_0/(\gamma_0 + \gamma_1)$ quantile of $f_{\Pi|T}(\pi|t; \alpha, \beta, n, p)$.

Further extensions of our stratified Bayesian model might include taking a hierarchical approach. This would allow the researcher to express a judgement about the similarities of $\pi_i$ and $\beta_i$ and to integrate these similarities into one model. The advantage of this approach is that the posterior distribution for each $\pi_i$ borrows strength from the likelihood contribution for all strata while reflecting our uncertainty about the true values of $\alpha$ and $\beta$. And possibly
recovering the loss of information associated with the use of randomization devices. For example, in the context of Section 4.2, one could add an additional level to the hierarchy by taking $z$ and $\beta$ to have exponential distributions. Insofar as implementation, adopting a hierarchical structure would not be prohibitive. For more information on hierarchical models, including computational issues, see Gelman et al. (2003).

Acknowledgements

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References