Measuring systemic risk using vine-copula

Armin Pourkhanalia, Jong-Min Kim, Laleh Tafakori, Farzad Alavi Fard

School of Economics, Finance and Marketing, RMIT University, Melbourne, VIC 3000, Australia
Division of Science and Mathematics, University of Minnesota at Morris, Morris, MN 56267, USA
School of Mathematics and Statistics, The University of Melbourne, Melbourne, VIC 3010, Australia

ARTICLE INFO

Article history:
Accepted 12 November 2015
Available online xxxx

Keywords:
Partial correlation
Vine copula
Credit risk

ABSTRACT

We present an intuitive model of systemic risk to analyse the complex interdependencies between different borrowers. We characterise systemic risk by the way that financial institutions are interconnected. Using their probability of default, we classify different international financial institutions into five rating groups. Then we use the state-of-the-art canonical (C-) and D-vine copulae to investigate the partial correlation structure between the rating groups. Amongst many interesting findings, we discover that the second tier financial institutions pay a larger contribution to the systemic risk than the top tier borrowers. Further, we discuss an application of our methodology for pricing credit derivative swaps.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

A major shortcoming in many of the models underlying the financial system is their inadequacy to fully comprehend the risk in extreme events. Not only is it essential that we concern ourselves with these unlikely events in isolation, or tail risk, but it is becoming increasingly evident that we should also concern ourselves with these unlikely events in tandem, or systemic risk. As the recent financial crisis illustrates, tail and systemic risk are very real and very devastating.

The copula is a mathematical tool for modelling the joint distribution of simultaneous events. From the perspective of tail and systemic risk, the copula is interesting in that it allows us to decouple the marginal distribution (that which is associated with tail risk) from the dependence structure (that which is associated with systemic risk) and model each separately with a greater degree of precision. Nevertheless, the problem in practical applications is how to identify this copula. For the bivariate case, a rich variety of copula families are available and well-investigated (see Brechmann et al., 2013; Brechmann and Joe, 2014). The use of copula is however challenging in higher dimensions, where standard multivariate copulae suffer from rather inflexible structures. Vine copulae overcome such limitations and are able to model complex dependency patterns by benefitting from the large range of bivariate copulae as building blocks.

1.1. Background

Since the global financial crisis (GFC) in 2008, there has been a strong interest in the development of accurate models for the dependency structure of default risk between financial institutions. The aftermath of the failure of Bear Stearns and Lehman Brothers, presented a significant systemic risk due to the interdependency of financial institutions across the globe. The impact of the global financial crisis reached its peak in Europe in July 2015, when Greece was placed in arrears on its public debt to the International Monetary Fund. This caused indexes worldwide to tumble, as many are now uncertain about Greece's future, fearing a potential exit from the European Union. Although current financial regulations have attempted to manage the systemic risk through the Basel capital requirements, their actions are micro-prudential in nature, in that they seek to limit each institutions risk. However, unless the external costs of systemic risk are internalized by each financial institution, the institution will have the incentive to take risks that are borne by others in the economy. Therefore, it is necessary for banks and other financial institutions to develop an active and market based management of credit risk.

It is known that the individual defaults do not have a significant impact on the risk of a bank's portfolio since they are (typically) well diversified. However, a portfolio of a large number of small loans, with systemic dependencies, is perceived to be very risky. To manage this risk, the conventional approach is to hedge the counterparty risk, using basket credit derivatives and collateralised debt obligations. As more financial firms try to manage their credit risk at the portfolio level, the demand for basket credit derivative products will most likely continue to grow.
Therefore, the problem of default correlation is central to the valuation of credit derivatives, even in the case of a simple credit default swap with one underlying reference asset. After the 1997–98 financial crisis in Southeast Asia,\footnote{Mansor et al. (2015) provides a comprehensive discussion on the 1997–98 Asian financial crisis and 2008–09 GFC, in the context of mutual fund performance.} which provided sufficient evidence of the default correlation between financial institutions and credit derivatives, a body of literature emerged recommending the use of Gaussian copula to model joint distribution of the probability of default. Li (1999) pioneered the use of Gaussian copula models for the pricing of collateralised debt obligations. The simplicity and the elegance of the approach soon drew attention in academia and amongst market practitioners in the valuation of credit derivatives. For example, in a comprehensive study, Das and Geng (2004) analysed the joint default process of hundreds of issuers, using different copula functions including normal, Gumbel, Clayton and Student $t$ copulae. In a more recent paper, J. Chen et al. (2014) employed a copula model with stochastic correlation for pricing credit derivatives portfolios, under the assumption that the systematic factor and idiosyncratic factors subject to the fat-tailed, mixed G-VG distribution instead of the traditional Gaussian distribution.

Copula models have also been used for modelling the dependency structure of other financial assets. For instance, Bhatti and Nguyen (2012) used the conditional extreme value theory and time-varying copulae to capture the tail dependence between selected international stock markets. Nalfar (2012) modelled the dependence structure between risk premium, equity return and volatility in the presence of jump-risk. Fenech et al. (2015) discussed loan default correlation using an Archimedean copula approach. Nguyen and Bhatti (2012) used nonparametric chiKendall plots and semi parametric copula to capture the dependency between oil prices and stock markets.

The popularity of copula models is due to the implication of Sklar’s theorem (see Sklar, 1959) that the modelling of the marginal distributions can be separated from the dependence modelling in terms of the copula. More importantly, copulae provide a convenient framework for measuring the extreme dependence between two random variables. In a crisis, financial correlations typically increase (see studies by Das et al. (2007) and Duffie et al. (2009)); hence, it would be desirable to apply a copula model with high co-movements in the lower tail of the joint distribution. However, the identification of the copula families for problems with higher dimensions than the bivariate cases remains as a major problem. Brechmann and Schepsmeier (2013) demonstrates that standard multivariate copulae such as the multivariate Gaussian or Student-$t$, as well as exchangeable Archimedean copulae lack the flexibility of accurately modelling the dependence amongst larger numbers of variables. Generalisations of these offer some improvement, but typically become rather intricate in their structure and hence exhibit other limitations such as parameter restrictions.

1.2. Vine copulae

To avoid all of these problems, in this paper we propose the use of vine copulae for more accurate modelling of the dependence amongst a larger number of variables. Vine copulae were initially proposed by Joe (1996) and further developed in Bedford and Cooke (2001), Bedford and Cooke (2002) as well as in Kurowicka and Cooke (2006). They use a cascade of bivariate copulae, known as pair-copulae, to build multivariate copulae, such that a multivariate probability density can be decomposed into bivariate copulae. Since each pair-copula is chosen independently, it provides a significantly more flexible framework for dependence modelling in credit risk. Most importantly, correlation asymmetries and tail dependencies can be taken into account to build more parsimonious models. In summary, vines combine the flexibility of bivariate copulae and the advantages of multivariate copula modelling, that is separation of marginal and dependence modelling.

Aas et al. (2009) introduced vine copulae into the finance and insurance literature, where they also described statistical inference techniques for the two classes of canonical C- (and D-) vines. Despite the material improvement that vine copulae provide, they have been rather overlooked in the credit risk literature. In this paper, for the first time we study the partial correlation structure by using vine copula. We propose that the partial correlation vine fully characterises the correlation structure of joint distribution, without having to assume independence between the variables (i.e., they may be related). Unlike the values in a correlation matrix, the partial correlation in a vine need not satisfy an algebraic constraint like positive definiteness. Kim et al. (2011) developed a robust statistical framework for the analysis of partial correlation with Gaussian vine copulae. In this paper, we take a similar approach, however, we also consider Gumbel, Clayton and student $t$ copula in joint dependence relationship, so that we can capture the stylised features of the financial data.

The remainder of this paper is structured as follows. In Section 2 we thoroughly describe the data on probability of default (PD) for financial institutions. Section 3 provides a discussion on empirical features of dependence in the joint distribution, which serves as the motivation for our choice of models. Section 4 provides the theoretical framework for vine copulae and Section 5 presents the empirical analysis and a thorough discussion on the model applications. Section 6 concludes the paper.

2. Data description

Our data set comprises global financial institutions, including banks and insurance companies, tracked by Thomson Reuters every month during the period of Jan 2005 to Jan 2015. For each issuer, we have obtained PDs based on Thomson Reuters’ structural model.

Thomson Reuters evaluates the equity market’s view of credit risk via a propitiatory structural default prediction framework based on the Merton model (Merton, 1974) which models a company’s equity as a call option on its assets. In this framework, the probability of default (PD) equates to the probability that the option expires worthless. Thomson Reuters produces daily updated estimates of the probability of default or bankruptcy within one year for 35,000 companies globally, including financials. The default probabilities are also mapped to letter ratings and ranked to create 1–100 percentile scores.

We consider 60 traded global financial institutions (banks and insurance companies), where we classify issuers into five credit rating groups. The PDs within these classes vary from high to low. Table 1 shows the sorting classification and Table 2 reports the descriptive statistics of our data from rating classes 1 through 5. In Table 2 we can observe that the mean and the standard deviation increase from the first rating class to the fifth rating class, where changes tend to be higher for lower-grade issuers. Table 3 presents the dependence between rating classes measured by Kendall’s $\tau$ statistic, a means of determining the dependence between any two time series. Fig. 1 pictorially illustrates the linear correlation of each rating classes, where each edge is based on weights of Kendall’s $\tau$ amongst two rating classes. Specifically, a higher correlation is presented by darker and thicker lines, and a lower correlation is presented by lighter and thinner lines (Fig. 1).

<table>
<thead>
<tr>
<th>Rating class</th>
<th>Rating</th>
<th>Credit group</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Aaa, Aa</td>
<td>High grade</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>High grade</td>
</tr>
<tr>
<td>3</td>
<td>Baa</td>
<td>Medium grade</td>
</tr>
<tr>
<td>4</td>
<td>Ba, B</td>
<td>Medium grade</td>
</tr>
<tr>
<td>5</td>
<td>Caa, Ca, C</td>
<td>Low grade</td>
</tr>
</tbody>
</table>

Table 1 Rating class the database comprises five rating categories.
Higher-grade issuers show more rank correlation in PDs than low-grade issuers. This is thoroughly discussed in the next section.

### 3. Empirical features of dependence in the joint distribution

In order to compare the statistical properties of joint distributions associated with different copulae, we develop a metric, which captures two crucial features of the dependence relationship in the joint distribution: tail-dependence and correlation asymmetry.

#### 3.1. Tail-dependence

Tail dependency is of significant importance when analysing the partial correlation structure for the default risk between different issuers. It has been well recognised that many financial assets exhibit a number of features which contradict the normality assumption, namely asymmetry, skewness and heavy tails. Moreover, asset return data also suggests a dependence structure which is quite different from the Gaussian (see Fortin and Kuzmics, 2002). Recent empirical studies, such as Junker and May (2005) and Malevergne and Sornette (2002), indicated that especially during highly volatile and bear markets the probability for joint extreme events leading to simultaneous losses in a portfolio could be seriously underestimated under the normality assumption. Theoretically, Embrechts et al. (2002) showed that the traditional dependence measure (the linear correlation coefficient) is not always suited for a proper understanding of the dependence in financial markets. When it comes to measuring the dependence between extreme losses other measures (e.g., the tail dependence coefficient) are more appropriate. This is particularly important in the credit risk framework, where the risk factors actually enter the model only to introduce a dependence structure in the portfolio. Frey and McNeil (2003) provided examples and insight on the impact of a violated Gaussian assumption on the tail of the credit portfolio loss distribution. Clearly, appropriate multivariate models suited for extreme events are needed.

The d-dimensional vine copulae are built via successive mixing from $d(d−1)/2$ bivariate linking copulae on trees of d nodes and their cumulative distribution functions involving lower-dimensional integrals. Since the densities of multivariate vine copulae can be factorised in terms of bivariate linking copulae and lower-dimensional margins, they are computationally tractable for high-dimensional continuous variables. Vine copulae cover a wide range of dependence and, as we will illustrate in this paper, by choosing bivariate linking copulae appropriately, vine copulae can have a flexible range of lower/upper tail dependence and different lower/upper tail dependence parameters for each bivariate margin. This has been discussed in great detail in Joe et al. (2010).

Table 4 reports the tail dependence between rating classes. We examine the correlation amongst observations in the bottom 10th, 20th, and 30th percentiles. We also present the correlations in the top 10th, 20th, and 30th percentiles (i.e., the cut-offs are the 70th, 80th, and 90th percentiles). The values in Table 4 represent the correlations when observations from the second rating in the pair of rating classes lies in the designated portion of the tail of the distribution. Results are presented for the levels of PD and for changes in the PD.

In subsection 4.2, we will discuss that the Gaussian, Clayton and Gumbel vine-copulae have one parameter, while the Student-t copula has two. The additional parameter of the latter is the degrees of freedom, controlling the strength of dependence in the tails of the bivariate distribution. The Student-t copula allows for joint extreme events, either in both bivariate tails or none of them. If one believes that the variables are only lower-tail dependent, a better choice might be the Clayton copula, because it exhibits greater dependence in the negative tail than in the positive. The Gumbel copula is also an asymmetric copula, but it exhibits greater dependence in the positive tail than in the negative.

#### 3.2. Asymmetric correlation

The correlation asymmetry in financial markets is well documented. Longin and Solnik (2001) are amongst the first ones to show the existence of asymmetry after controlling for the bias coming from the conditioning. Ang and Chen (2002) makes a similar observation, when looking at the correlations between U.S. stocks and the aggregate U.S. stock market conditional on downside and upside moves. Hong et al. (2007) and Buraschi et al. (2010) also found that incorporating the asymmetry in the portfolio choice decision bring significant gains.

In the context of credit risk, this phenomenon has a further intuitive appeal since it indicates that high graded issuers (in many cases large firms) have a larger exposure to systemic risk, whereas low graded issuers (medium and small firms) face more idiosyncratic risk (see Das and Geng, 2004). This paper includes the correlation asymmetry in the modelling framework, according to Ang and Chen (2002) and Longin and Solnik (2001), where we consider the exceedance correlation measure, as follows.

We collect the PDs across all financial institutions at each point in time, $t$, then we normalize them by subtracting the mean from each observation and dividing them by the standard deviation. For an exceedance level at quantile $q$, for pairs of standardized observations $(PD_a, PD_b)$, we select a subset of observations such that

$$\{(PD_a, PD_b) | PD_a < q, PD_b < q\},$$

and

$$\{(PD_a, PD_b) | PD_a > 1 - q, PD_b > 1 - q\}.$$
Furthermore, we calculate both \( \text{corr}(PD_{it}, PD_{jt}) | PD_{it} > 0.5, PD_{jt} > 0.5) \) and \( \text{corr}(PD_{it}, PD_{jt}) | PD_{it} < 0.5, PD_{jt} < 0.5) \) when \( q = 0.5 \). Therefore, the exceedance correlation \( \rho^- \) is lower \( q \) quantile, while \( \rho^+ \) refers to the joint occurrence of positive changes, above \( 1 - q \). Thus,

\[
\rho^-(q, =, =) = \text{corr}(PD_{it}, PD_{jt} | PD_{it} > 1-q, PD_{jt} > 1-q),
\]

\[
\rho^-(-\infty, q, -\infty) = \text{corr}(PD_{it}, PD_{jt} | PD_{it} < q, PD_{jt} < q).
\]

Here \( i \) and \( j \) denote any two different rating classes. The correlations for different rating classes are demonstrated in Fig. 3. The height of the exceedance line indicates the level of correlation, where we can see that high grade issuers have greater correlation than lower grade issuers. Additionally, there is clear evidence of correlation, where we can see that high grade issuers have greater correlation. Tail dependency of corporations’ PDs have been discussed in the literature in the context of systematic risk before; however, we reason that the argument might not be accurate. Most prominently, Das and Geng (2004) argued that the upper-tail dependency in higher grade firms is associated with their higher exposure to the systematic risk, while lower grade firms evidence more idiosyncratic risk and, hence, exhibit a lesser degree of upper-tail dependency. Nevertheless, the systematic risk is exogenous to this modelling framework, which, coupled with the absence of Granger causality, impedes a valid inference. Additionally, for some credit groups in Fig. 3 we observe a high level of lower-tail dependency which, although derived from a different data set, somehow contradicts the results of Das and Geng (2004), thus their argument does not universally hold.

It is more accurate to explain the tail dependency between credit risk of different issuers in the context of systemic risk (aka the spillover risk). Financial institutions hold a portfolio of fixed income securities issued by other financial institutions. However, due to their capital adequacy requirements, they tend to overweight their exposures to high grade issuers. Inherently, the upper-tail dependency of all credit groups, conditioned to the top two groups are high, indicating that the system is more vulnerable to the failure of higher graded issuers. Bank failure contagion occurs faster within these issuers, spreads more broadly within the industry, and results in a larger number of failures, as well as larger losses to creditors at failed banks. Whereas, the failure of low graded financial institutions is typically absorbed by the system, evidenced by lesser tail dependency of all rating groups conditioned to group 5, in Fig. 3.

The partial correlation structure between the PDs of different credit groups measures the interdependency between the average credit of the constituents of each group, hence it is a measure of systemic risk. We use vine-copula to accurately quantify this measure of systemic risk, which, by way of construction, better models the interdependency than the popular pair-copula.

### 4. Copula functions and vine copula

A copula is a multivariate uniform distribution representing a way of trying to extract the dependence structure of the random variables from the joint distribution function. It is a useful approach to understanding and modelling dependent random variables. Let \( X_i, i \in \{1, \ldots, r\} \), to be random variables, denoting the average probability of default in each credit rating group. Here \( i \) is the index number of the credit group, where the ascending order of \( i \) corresponds to the descending order credit quality. In this paper, we consider the total of 5 credit groups \( (r = 5) \), then \( X_i \) is the average PD for the highest quality issuers and \( X_6 \) for the lowest.

Further, let the dependence structure of \( X_i \) be contained within \( H \). The idea of separating \( H \) into one part which describes the dependence structure and other parts which describe only the marginal behaviour has led to the concept of a copula. Every joint distribution can be written as per the following.
Definition 4.1. A r-dimensional copula is a function \( C : [0, 1]^r \to [0, 1] \) with the following properties:

1. For all \((u_1, \ldots, u_r) \in [0, 1]^r\), then \( C(u_1, \ldots, u_r) = 0 \) if at least one coordinate of \((u_1, \ldots, u_r)\) is 0;
2. \( C(1, \ldots, u_i, 1, \ldots, 1) = u_i\) for all \( u_i \in [0, 1] \), \((i = 1, \ldots, r)\);
3. \( C\) is r-increasing. (see Nelsen, 2013, Definition 2.10.2).

Sklar’s theorem clarifies the role that copulas play in the relationship between multivariate distribution functions and their univariate margins.

Theorem 4.1. Sklar’s theorem

Let \( H \) be a joint distribution function with margins \( F \) and \( G \). Then there exists a copula \( C \) such that for all \( x, y \in \mathbb{R} \),

\[
H(x, y) = C(F(x), G(y)).
\]

If \( F \) and \( G \) are continuous, then \( C \) is unique; otherwise, \( C \) is uniquely determined on \( \text{Ran} F \times \text{Ran} G \). Conversely, if \( C \) is a copula and \( F \) and \( G \) are distribution functions, then the function \( H \) defined by (1) is a joint distribution function with margins \( F \) and \( G \) (Nelsen, 2013).

A copula is thus a function that, when applied to marginal distributions, results in a proper multivariate probability distribution function. Since this density function embodies all the information about the random vector, it contains all the information about the dependence structure of its components. Hence by implementing this technique, we split the distribution of a random vector into individual components (marginal) with a dependence structure (the copula) without losing any information.

Further, let \( X \) and \( Y \) be random variables with continuous cumulative distribution functions (CDF) \( F(x) \) and \( G(y) \), respectively, let \( X \) and \( Y \) be continuous random variables with copula \( C \) and marginal distribution functions \( F(x) \) and \( G(y) \), so that \( X \sim F(x), \ Y \sim G(y), \) and \( (X, Y) \sim H_{XY}(x, y) \). Let \( u = F(x), v = G(y), \) and \((u, v) \sim C\). The coefficients of upper and lower tail dependence of \((X, Y)\) are defined by (Nelsen, 2013) as

\[
\tau_U = \lim_{\epsilon \to 0} P(u \leq \epsilon | v \leq \epsilon) = \lim_{\epsilon \to 0} P(v \leq \epsilon | u \leq \epsilon),
\]

\[
\tau_L = \lim_{\delta \to 1} P(u > \delta | v > \delta) = \lim_{\delta \to 1} P(v > \delta | u > \delta),
\]

if \( \tau_U \in [0, 1] \) and \( \tau_L \in [0, 1] \) exist. Here, \( \tau_U (\tau_L) \) is the coefficient of lower- (upper-) tail dependence.

Fig. 3. Asymmetric correlation for every pair of rating classes in the data.
Table 5 presents the closed form expressions for $\tau^a$ and $\tau^b$. In the table, $\rho$ is the coefficient of linear correlation of two random variables, $\nu$ is the degree of freedom, and $\alpha$ and $\beta$ are dependence parameters. Additionally, $\tau^a$ and $\tau^b$ for the Gaussian Copula are zero if and only if $\rho < 1$.

**Fig. 6** pictorially presents the densities of the four copulae for three different parameter specifications.

### 4.1. Vine copula

Vine copula is a graphical model, proposed in Bedford and Cooke (2001), Bedford and Cooke (2002) and Cooke (1997). A vine on $n$ variables is a nested set of trees, where the edges of tree $i$ are the nodes of tree $i + 1$, and each tree has maximum number of edges. The trees are called dependence vines, which are used to specify dependence structures in high-dimensional distributions.

**Definition 4.2.** $\nu$ is vine on $n$ elements if

1. $v = (T_1, \ldots, T_m);$
2. $T_1$ is a connected tree with nodes $N_1 = \{1, \ldots, n\}$ and edges $E_1;$
3. for $i = 2, \ldots, m$, $T_i$ is connected tree with nodes $N_i \subset N_{i-1} \cup E_{i-1} \cup \cdots \cup E_1 - 1$. 

**Definition 4.3.** A vine $v$ is a vine on $n$ elements if

1. $m = n$, 
2. for $i = 2, \ldots, n - 1$, if $\{a, b\} \in E_i$, then $a \neq b$, where $\Delta$ denotes the symmetric difference. In other words, if $a$ and $b$ are nodes of $T_i$ connected by an edge in $T_i$, then exactly one of the $\{a, b\}$ equals one of the $\{a, b\}$.

A more formal definition of vines can be found in (Bedford and Cooke, 2002). There are only two types of vines which are of interest in this paper, namely D-vine and Canonical (C-vine) which are defined as follows:

**Definition 4.4.** D-vine

If each node in $T_i$ has a degree of at least 2, then the vine is a D-vine.

**Definition 4.5.** Canonical or C-vine

If each tree $T_i$ has a unique node of degree $n - i$, then the vine is a canonical vine. The node with maximal degree in $T_1$ is the root (Kurowicka and Cooke, 2006).

In C-vine copula each tree $T_i$ has a unique node that is connected to all other nodes and in D-vine copula each tree is a path and each of them give a special way to decompose the density. The specification may be given in the form of a nested set of trees. From the application point of view, D-vine is more effective for temporal ordering of variables, while C-vine is more useful for ordering by importance.

The number of different D- and C-vines is very large. Aas et al. (2009) showed that for a C-vine decomposition on $n$ nodes there are $1/2$ distinct C-vine trees, which is also the number of distinct D-vine trees. This means that we need additional structure to select reasonable vine trees. At first it may be reasonable to restrict to C- and D-vine trees, by choosing a variable which drives all other variables. For this paper, we consider the rating groups “1” and “2” as the driving variable for the systematic risk. Both rating groups “1” and “2” contain financial institutions with the largest market capitalisation, which tend to have lower capital, less-stable funding, more market-based activities, and be more organisationally complex than small financial institutions. This suggests that the failures of large financial institutions generate liquidity stress in the system. Further, their activities that rely on economies of scale and scope cannot easily be replaced by small financial institutions and the marginal cost of taxpayer support may increase in the volume required.

For choosing between the rating groups “1” and “2”, as well as the order in the trees, we consider the second criterion suggested by Aas et al. (2009), that is: put the strongest bivariate dependencies in the first tree of the vine tree specification. Strongest bivariate dependencies within the copula distribution can be measured by Kendall’s $\tau$ or the tail dependence coefficient, which we have reported in Table 3 (other approaches have also been recommended in the literature, such as the smallest partial correlation method of Lanzendörfer and Min, 2009).

The trees in Fig. 4 show the specification corresponding to a five-dimensional D-vine and C-vine copula. They each consist of four trees $T_i$ where $i = 1, \ldots, 4$. Each edge corresponds to a pair-copula density and the edge label corresponds to the subscript of the pair-copula density. The nodes in tree $T_i$ are only necessary for determining the labels of the edges in tree $T_i + 1$. The numbers 1 to 5 of nodes in the first order trees represent the rating groups from the highest to the lowest, respectively. This presentation of vine-copulae is similar to the conventions introduced in (Bedford and Cooke, 2001; Bedford and Cooke, 2002).

### 4.2. Partial correlation and conditional tail dependence by multivariate copulae

Bedford and Cooke (2001) showed that any assignment of values in the open interval $(-1, 1)$ to the edges in any partial correlation vine is consistent, the assignments are algebraically independent, and there is a one-to-one relationship between all such assignments and the set of correlation matrices. In other words, partial correlation vines provide an algebraically independent parameterisation of the set of correlation matrices, whose terms have an intuitive interpretation. Moreover, the determinant of the correlation matrix is the product over the edges of $(1 - R^2_{ik,k|k})$ where $R^2_{ik,k|k}$ is the partial correlation assigned to the edge with conditioned variables $i$, $k$ and conditioning variables $D(ik)$. A similar decomposition characterises the mutual information, which generalises the determinant of the correlation matrix.

Built on the model proposed in (Kim et al., 2011), we model the correlation structure of PDs for different credit groups using vine-copula partial correlation. Kim et al. (2011) analyses the partial correlation using only the Gaussian copula, however, to achieve a more realistic model for financial data we also extend the analysis to Student $t$, Clayton and Gumbel copulæ. This becomes particularly important, when analysing the tail dependencies.
Given an $n$-dimensional distribution function $F$ with continuous marginal (cumulative) distributions $F_{1}, \ldots, F_{n}$, there exists a unique $n$-copula $C : [0, 1]^{n} \rightarrow [0, 1]$ such that

$$F(x_{1}, \ldots, x_{n}) = C(F_{1}(x_{1}), \ldots, F_{n}(x_{n})).$$

Suppose $Y$ and $Z$ are real-valued random variables with conditional distribution functions $F_{2|1}(y|x) = P(Y \leq y|X = x)$, and

$$F_{3|1}(z|x) = P(Z \leq z|X = x).$$

Then the basic property of $U = F_{2|1}(Y|X)$ and $V = F_{3|1}(Z|X)$ is as follows: suppose, for all $x$, $F_{2|1}(y|x)$ is continuous in $y$ and $F_{3|1}(z|x)$ is continuous in $z$. Then $U$ and $V$ have uniform marginal distributions. Likewise, if $X_{1}, \ldots, X_{n}$ is a vector of $n$ random variables with absolutely continuous multivariate distribution function $F$, then the $n$ random variables

$$U_{1} = F_{1}(X_{1}), U_{2} = F_{2}(X_{2}|X_{1}), \ldots, U_{n} = F_{n}(X_{n}|X_{1}, \ldots, X_{n-1}),$$

are i.i.d. $U(0, 1)$.

The conditional distribution of $Z_{2}$ given $Z_{1}$ is also normal with mean vector

$$v_{1} = \mu_{1} + \Sigma_{12} \Sigma_{22}^{-1}(z_{2} - \mu_{2}),$$

and covariance matrix

$$Q_{1} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}. $$

It follows that the conditional density function $f_{12}(\cdot|Z_{2})$ of $Z_{1}$, when $Z_{2} = z_{2}$, is specified at the point $z_{1}$ by the equation

$$f_{12}(z_{1}|Z_{2}) = \frac{f(z_{1}, z_{2})}{f(z_{2})} = \frac{1}{2 \pi^{\nu/2} |\Sigma|^{1/2}} \exp \left\{ - \frac{1}{2} (z_{1} - \mu_{1}) \Sigma^{-1} (z_{1} - \mu_{1}) \right\}.$$

The cumulative distribution function is

$$F_{12}(z_{1}|Z_{2}) = \int_{-\infty}^{z_{1}} \int_{-\infty}^{z_{2}} f(x_{1}, x_{2}) dx_{1} \cdots dx_{p}, \quad (4)$$

where $z_{1} = (z_{1}, \ldots, z_{p})$ and $z_{1}, \ldots, z_{p} \in \mathbb{R}$.

A conditional joint cumulative distribution $H$ expressed as a function of the conditional margins $F_{G}$ is

$$H(x, y|z; \theta_{1}, \theta_{2}, \alpha) = C(F(x|z; \theta_{1}), G(y|z; \theta_{2}); \alpha), \quad (5)$$

and its joint probability density function is

$$h(x, y|z; \theta_{1}, \theta_{2}, \alpha) = c(F(x|z; \theta_{1}), G(y|z; \theta_{2}); \alpha)f(x|z; \theta_{1})g(y|z; \theta_{2}).$$

4.2.1. The multivariate Gaussian copula

By using Eq. (2), we can derive the gaussian conditional distributions, and then by using the CML method by Genest et al. (1995) and the IFM method by Joe (1997), we can estimate the gaussian copula parameter, a $n$-th order conditional correlation, $\rho_{YX|Z_{1}, Z_{2}, \ldots, Z_{n}}$, using the following:

$$F_{0}(y_{1}|X|Z_{1}, Z_{2}, \ldots, Z_{n}) = C_{x}^{0}(F_{x_{1}|Z_{1}, \ldots, Z_{n}}(X|Z_{1}, Z_{2}, \ldots, Z_{n})), \quad F_{y_{1}|Z_{1}(Y|Z_{1}, Z_{2}, \ldots, Z_{n}), \rho_{YX|Z_{1}, Z_{2}, \ldots, Z_{n}}.}$$

In this paper, we used the IFM method by Joe (1997) to estimate the Gaussian copula partial correlation parameter, $\rho_{YX|Z_{1}, Z_{2}, \ldots, Z_{n}}$.

4.2.2. The multivariate Student-$t$ distribution

The $n$-dimensional random vector $X = (X_{1}, \ldots, X_{n})$ is said to have a (non-singular) multivariate Student-$t$ distribution with $v$ degrees of freedom, mean vector $\mu$ and positive-definite dispersion or scatter matrix $\Sigma$, denoted $X \sim t_{v}(\mu, \Sigma)$, if its density is given by

$$f(x) = \frac{\Gamma \left( \frac{v + n}{2} \right)}{\Gamma \left( \frac{v}{2} \right)} \left( 1 + \frac{\mathbf{1}^{\top} x - \mu \Sigma^{-1} (x - \mu) \mathbf{1}}{v} \right)^{-\frac{n}{2}}.$$
4.2.4. Multivariate Gumbel copula

The multivariate n-variate Gumbel copula is:

\[ C^G(u_1, u_2, \ldots, u_n) = \varphi^{-1}\left[ \sum_{i=1}^{n} \varphi(u_i) \right], \]

where \( \varphi \) is a function from \([0, 1]\) to \([0, \infty)\) such that

(i) \( \varphi \) is a continuous strictly decreasing function,

(ii) \( \varphi(0) = \infty \) and \( \varphi(1) = 0 \),

(iii) \( \varphi^{-1} \) is completely monotonic on \([0, \infty)\).

If the generator is given by \( \varphi(u) = u^{-\alpha} - 1 \), then we get the multivariate n-variate Gumbel copula as follows:

\[ C^G(u_1, u_2, \ldots, u_n) = \left[ \sum_{i=1}^{n} u_i^{-\alpha} - n + 1 \right]^{-1/\alpha}, \quad \text{with} \ \alpha > 0. \]

4.2.4. Multivariate Gumbel copula

The generator is given by \( \varphi(u) = (-\ln(u))^\alpha \), hence \( \varphi^{-1}(t) = \exp(-t^{1/\alpha}) \); it is completely monotonic if \( \alpha > 1 \). The multivariate n-variate Gumbel copula is given by

\[ C^G(u_1, u_2, \ldots, u_n) = \exp\left\{ -\left[ \sum_{i=1}^{n} (-\ln u_i)^2 \right]^{1/\alpha} \right\}, \quad \text{with} \ \alpha > 1. \]

**Corollary 1.** Suppose we have three normal random variables \( X_1, X_2, X_3 \) and we let \( U_1 = F_{X_1}(X_2) \) and \( U_2 = F_{X_2}(X_3) \). Using one of Archimedean copulas, Clayton Copula (Clayton, 1978)

\[ C_\alpha(u_1, u_2) = \left( u_1^{\alpha_2} + u_2^{\alpha_2} - 1 \right)^{-1/\alpha_2}, \]

for \( \alpha_2 > 0 \), we can derive the following ones:

\[ C^\alpha(u_2, u_3) = \left( u_2^{\alpha_1} + u_3^{\alpha_2} - 1 \right)^{-1/\alpha_1}, \]

\[ C^\alpha(u_1, u_2, u_3) = \left( u_1^{\alpha_1} + u_2^{\alpha_2} + u_3^{\alpha_3} - 2 \right)^{-1/\alpha_1}, \]

\[ \frac{\partial^2 C^\alpha(u_2, u_3)}{\partial u_2 \partial u_3} = -\frac{1}{\alpha_1 \alpha_2} \left( u_2^{\alpha_1} + u_3^{\alpha_2} - 1 \right)^{1/\alpha_1 \alpha_2} \times \left( u_2^{\alpha_1} - \alpha_1 u_2^{\alpha_1 - 1} - \alpha_2 u_3^{\alpha_2 - 1} \right), \]

and

\[ \frac{\partial^2 C^\alpha(u_1, u_2, u_3)}{\partial u_2 \partial u_3} = -\frac{1}{\alpha_1 \alpha_2 \alpha_3} \left( u_1^{\alpha_1} + u_2^{\alpha_2} + u_3^{\alpha_3} - 2 \right)^{1/\alpha_1 \alpha_2 \alpha_3} \times \left( u_1^{\alpha_1} - \alpha_1 u_1^{\alpha_1 - 1} - \alpha_2 u_2^{\alpha_2 - 1} - \alpha_3 u_3^{\alpha_3 - 1} \right). \]

We can derive the conditional distributions \( F_{1|23}(X_1|X_2, X_3) \) and \( F_{4|23}(X_4|X_2, X_3) \) by Corollary 1 as follows:

\[ F_{1|23}(X_1|X_2, X_3) = C^\alpha_1(u_1, u_2, u_3) = \frac{P[U_1 \leq u_1 | U_2 = u_2, U_3 = u_3]}{\frac{\partial^2 C^\alpha(u_1, u_2, u_3)}{\partial u_2 \partial u_3} \left/ \frac{\partial^2 C^\alpha(u_2, u_3)}{\partial u_2 \partial u_3} \right.} \]

similarly.

\[ F_{4|23}(X_4|X_2, X_3) = C^\alpha_4(u_4|u_2, u_3) = \frac{P[U_4 \leq u_4 | U_2 = u_2, U_3 = u_3]}{\frac{\partial^2 C^\alpha(u_4, u_2, u_3)}{\partial u_2 \partial u_3} \left/ \frac{\partial^2 C^\alpha(u_2, u_3)}{\partial u_2 \partial u_3} \right.} \]

where \( u_1 = F(X_1), u_2 = F(X_2), u_3 = F(X_3) \) and \( u_4 = F(X_4) \).

Since \( F_{1|23}(X_1|X_2, X_3) \) and \( F_{4|23}(X_4|X_2, X_3) \) are i.i.d. \( U(0, 1) \), we can derive the conditional cumulative distribution function as follows:

\[ F_{1|23}(u_1|u_2, u_3) = C^\alpha_1(F_{1|23}(X_1|X_2, X_3), F_{4|23}(X_4|X_2, X_3), \alpha_1 \alpha_2 \alpha_3). \]

Similarly, the conditional cumulative distribution function by the multivariate n-variate Gumbel copula can be derived by Corollary 1. By using Eqs. 4 and 5, we can compute the coefficients of conditional
Step 2 The parameters

Step 1 Uses the empirical CDF to transform the observations to uniform cumulative distribution function can be summarised by:

\[ F(x|y) = \frac{F(x)}{1-F(y)} \]

Step 3 The parameters \( \theta_{ij} \) of the conditional CDF’s \( F(x_i|x_j) \) and \( F(x_j|x_i) \) are estimated by the IFM method by (Joe, 1997).

Step 4 The parameter \( \theta_{ijk} \) of the conditional CDF’s \( H(x_i|x_j|x_k) \) are estimated by the IFM method by (Joe, 1997).

Step 5 Compute the coefficients of conditional copula upper and lower tail dependence by using Eqs. (4) and (5).

5. Empirical results and applications

For the data described in Section 2, we evaluate the correlation structure of the PDs for 60 financial institutions, classified in 5 rating groups. In Subsection 4.1 we discussed the construction of C- and D-vine copulae, as well as the economic intuition for our choice of the driving variable (the rating group “2”). Here, we provide the parameter estimations and discuss the results. We have used the R package ‘VineCopula’ for the parameterisation of copulae models, and have used “Uncertainty analysis with Correlations” (UNICORN) software for the sampling and the graphical presentations for the partial correlations. UNICORN contains a graphical feature that enables an interactive visualisation of a moderately high-dimensional distribution (Kurowicka and Cooke, 2006). Here, we have five random variables (i.e., credit rating groups), which are transformed to uniform distribution, as per the algorithm discussed in the previous section. Fig. 5 plots the uniform transformed data. They depict that the dependence becomes clear when we transform the five variables to ranks or percentiles.

Table 6 provides partial correlation by using C-vine and D-vine Gaussian copulae, as well as the tail dependencies for four copulae that have different strength of dependence in the tails of the multivariate distribution; the Gaussian, Student-t, Clayton and Gumbel copulae. The first two are copulae of normal mixture distributions. They are so-called implicit copulae because they do not have a simple closed form. Clayton and Gumbel are Archimedean copulae, for which the distribution function has a simple closed form. The Clayton copula is lower-tail dependent, but not upper. The Gumbel copula is upper-tail dependent, but not lower. The Student-t copula is both lower- and upper-tail dependent, while the Gaussian is neither lower- nor upper-tail dependent (see Aas et al., 2009 for the detailed discussion).

Our empirical results are consistent with the theory above. As discussed in Section 2, our data set features a strong upper-tail dependency and a weaker lower-tail dependency. Similarly, in Table 6 we can see that the tail dependency is best captured by the Gumbel copula, and Clayton provides the second best fit. Further, neither the Student, nor the Gaussian Capulae provide any significant results, when used to capture the tails dependency. To illustrate the behaviour of the densities of Clayton and Gumble copulae, we plot the 3D bivariate densities in Fig. 6 using the three different estimated parameters reported in Table 6.

From the practical point of view this results are significant. For example consider first-to-default (FD) and last-to-default (LD), two very popular basket default swap products. For a FD default swap, whenever an entity in the reference basket defaults, the buyer stops paying the swap’s premium and receives from the seller the difference of the principal amount of the defaulted entity and the recovered value. If the swap’s counterparty defaults, premium payments will stop and both the buyer and the seller walk away from the contract. A LD default
swap works in a same way, but for the premium is calculated based on the principal amount of the latest defaulted entity and the recovered value.

Hu and Kercheval (2008) showed that a copula function with lower tail dependence (e.g., Clayton) leads to the highest default probabilities for LD, while a copula function with upper tail dependence (Gumbel copula) leads to the lowest default probabilities. The tail dependent Student-t copula leads to higher default probabilities than tail independent Gaussian copula. Further, the Clayton copula with only lower tail dependence leads to the lowest FD probabilities, while the Gumbel copula with only upper tail dependence leads to the highest FD probabilities.

Further, default events tend to happen when the uniform random variables \( U \) are close to 0. Since the LD requires that both uniform variables in the basket are small, a lower tail dependent copula will lead to higher LD probabilities than a copula without lower tail dependence.

Table 6 also reports the partial correlations between different rating groups. The first order partial correlations in both C- and D-vine trees reveal the importance of the rating group “2” in the systemic risk. This result is consistent with the findings of (Das and Sundaram, 2000). It is also important to highlight the importance of correctly selecting the tree structure, in particular for the C-vine copula. As discussed in Subsection 4.1, we used the method prescribed by Aas et al. (2009) for choosing the orders of the tree. Now suppose due to an error, an analyst chooses the rating group “1” as the driving variable for the systemic risk. This is a plausible mistake, in particular because the average Kendall \( \tau \) of rating groups “1” and “2”, reported in Table 3, are very close, as well as the fact that rating group “1” contains the so called too-big-to-fail firms. Under this scenario, the first order partial correlation (we use \( \hat{\tau} \) instead of \( \tau \) to indicate the parameters under the analyst’s incorrect model constructs) would have been

\[ \hat{\tau}_{23|1} = 0.803 \quad \hat{\tau}_{24|1} = 0.643 \quad \hat{\tau}_{25|1} = 0.445. \]

This would be clearly incorrect, since a) the results strongly contradict the D-vine results and b) the correlation between the other rating classes after eliminating the influence of the rating group “1” would be too strong. The latter would be inconsistent with the empirical observations in the financial market.

![Cobweb plot by Unicorn](image1)

![Parallel plot by Unicorn](image2)

Fig. 7. (a): Five different PD rating classes (uniform transformed data) cobweb plot by Unicorn. (b): Five different PD rating classes (uniform transformed data) parallel plot by Unicorn.
Table 6
This table reports the Gaussian Copula partial correlations between the five different rating classes, as well as the tail dependencies using t-Copula, Clayton and Gumbel. The parameters are estimated for both canonical vine (top panel) and D-vine (bottom panel).

<table>
<thead>
<tr>
<th>C-vine</th>
<th>Partial correlations</th>
<th>Tail dependencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>( r_{13(12)} )</td>
<td>( \tau_{13(12)} )</td>
</tr>
<tr>
<td></td>
<td>-0.393</td>
<td>0.0010</td>
</tr>
<tr>
<td></td>
<td>-0.263</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>-0.179</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.589</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.445</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.508</td>
<td>0.0123</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>D-vine</th>
<th>Partial correlations</th>
<th>Tail dependencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( r_{13(12)} )</td>
<td>( \tau_{13(12)} )</td>
</tr>
<tr>
<td></td>
<td>-0.391</td>
<td>0.0010</td>
</tr>
<tr>
<td></td>
<td>0.066</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.177</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.088</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>-0.052</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>-0.263</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>0.0123</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Fig. 7 provides the cobweb plot for the dependency structure between the credit rating groups. Panel (a), represents the possible values ining classes, as well as the tail dependencies using Table 6

5.1. Systemic risk and credit default swaps spread
As an important application, consider portfolio credit derivatives, such as basket credit default swaps (basket CDS). They require for their pricing an estimation of the dependence structure of defaults, which is known to exhibit tail dependence as reflected in observed default contagion. Without going in to the details of the pricing methodology, here we discuss some applications for our modelling set-up in this context.

A CDS is a contract that provides insurance against the risk of default of a particular company. The buyer of a CDS contract obtains the right to sell a particular bond issued by the company for its par value once a default occurs. The buyer pays to the seller a periodic payment, as a fraction of the nominal value, until the maturity of the contract or until a default at time occurs. If a default occurs, the buyer still needs to pay the accrued payment from the last payment time to the default time.

Since the payoff of a CDS contract is caused by the default on debt, CDS spreads have become a popular market-based indicator of the creditworthiness of the reference firm. When the CDS spreads widen, it indicates that the market participant’s expectations of default by the reference firm has grown. Some early evidence of this relationship may be found in (Hull et al., 2004; Norden and Weber, 2004), where they documented that the CDS market anticipates later rating announcements by the credit rating agencies.

In terms of financial contagion, there is a clear relationship between CDS and systemic risk: If there is a systemic event in the market, default expectations of relevant institutions should rise, which is then reflected in increasing CDS spreads. Because CDS contracts involve counterparty risk, this is reflected in their prices. Then, the set of prices of all CDSs written by each member of the financial network on the other members, together with bond prices, reflects the risk-neutral probabilities of default of each institution and of each pair of institutions in the network. Therefore, several researchers have developed measures of systemic risk that are based on CDS spreads (see for instance Huang et al., 2009; Giglio, 2011). CDS spreads have also been used to examine interdependencies amongst financial institutions: see Markose et al. (2012), Kaushik and Battiston (2012) and H. Chen et al. (2014). None of the authors, however, use copulae to account for non-standard interdependencies amongst the institutions.

Brechmann et al. (2013) and Brechmann and Joe (2014) demonstrated that the heterogeneous dependencies in the financial sector cannot be appropriately captured using an Archimedean copula, which assumes exchangeability of all variables. While elliptical copulae are more appropriate for this purpose, they are still somewhat restrictive by imposing symmetric tail dependence. In the literature, it is however often observed that in times of crisis the dependence of joint negative events increases. For CDS spreads, this means that one may expect the presence of upper tail dependence, which reflects the joint probability of extreme upward jumps in the expected default probabilities. Such dependence characteristics can be accounted for using a vine copula.

As a result analysis of systemic risk and pricing and hedging CDSs are complementary to each other. A summary of the steps involved in pricing CDS with vine-copulae is provided below. A detailed discussion on the mathematical properties of the pricing methodology, in particular when vine-copulae is used, may motivate future research in this domain.

1. Identify the dependence structure of the default times, using firm-specific critical variables.
2. Calibrate the vine-copula from a range of copula families, as discussed in Section 4.
3. Separately calibrate the default intensities of each firm using a reduced form model. See Elliott et al. (2000) for a discussion on different reduced form default risk models.
4. Use the default intensities to calculate survival functions for each of the firms.
5. Using the vine-copulae from Step 2, develop the distribution of n-to-default times by Monte Carlo sampling of many scenarios. For computationally intensive applications, UNICORN is a fast and efficient computer package which can assist with the simulation, based on the algorithm in (Cooke, 1997). Alternatively, most recently (Cooke et al., 2015) has introduced an improved algorithm for sampling, conditionalising and simulating regular vines.
6. Use these distributions to compute the risk neutral expectations in the pricing formulae.

6. Conclusions
In this paper, we put to work the recently developed C-vine and D-vine copulae for the analysis of interdependencies amongst financial institutions for systemic risk measurement. The application of copulae is often more art than science. There will generally never be that one obviously correct answer; however, there are often many wrong answers. More specifically, there are many copula structures which fail to adequately account for the behaviour in the extreme tails of univariate and multivariate loss distributions and as such greatly underestimate the tail and systemic risk. In this paper, we have highlighted several considerations with regard to more appropriately capturing both the tail and systemic risk. We analyse the dependence structure amongst 60 international financial institutions, classified into five rating classes. Further, we measured the partial correlation of the rating classes between Jan 2005 to Jan 2015, which includes the global financial crisis, and the following European debt crisis. We estimate the parameters of conditional CDF for different copulae, from which we parameterised the C-vine and D-vine copulae and tail dependency for the observations.

Further, we have provided a thorough discussion of the empirical findings, as well as the applications of our model for pricing CDSs. Amongst many interesting findings, we discover that the second tier financial institutions pay a larger contribution to the systemic risk than the top tier borrowers.
Acknowledgements

The authors would like to thank the editor and two anonymous referees for their constructive suggestions that greatly improved the paper. The third author would like to thank the Australian Research Council for supporting this work through Laureate Fellowship FL130100039.

References


Joe, H., 1996. Families of m-variate distributions with given margins and m (m−1)/2 bivariate dependence parameters. Lecture Notes-Monograph Series pp. 120–141.


Kaushik, R., Battiston, S., 2012. Credit Default Swaps: Drawdown Networks: Too Tied To Be Stable?


