Generalized bivariate copulas and their properties

Jong-Min Kim\textsuperscript{a,*}, Engin A. Sungur\textsuperscript{a}, Taeryon Choi\textsuperscript{b} and Tae-Young Heo\textsuperscript{c}

\textsuperscript{a}Statistics Discipline, Division of Science and Mathematics, University of Minnesota, Morris, MN, USA
\textsuperscript{b}Department of Statistics, Korea University, Seoul, 136-701, South Korea
\textsuperscript{c}Department of Data Information Science, Korea Maritime University, Busan, 606-791, South Korea

Abstract. Copulas are useful devices to explain the dependence structure among variables by eliminating the influence of marginals. In this paper, we propose a new class of bivariate copulas to quantify dependency and incorporate it into various iterated copula families. We investigate properties of the new class of bivariate copulas and derive the measure of association, such as Spearman’s $\rho$, Kendall’s $\tau$, and the regression function for the new class. We also provide the concept of directional dependence in bivariate regression setting by using copulas.

Keywords: Bivariate copulas, Spearman’s $\rho$, Kendall’s $\tau$, Marginal distribution, Regression function, Farlie-Gumbel-Morgenstern copula, Directional dependence

1. Introduction

It is becoming increasingly important and useful to be able to model dependence mechanisms among variables. Broadly speaking, two following approaches can be considered: (i) setting up a functional relation among the variables, and (ii) specifying the joint distribution of the variables. Copulas, introduced by Sklar [14], are useful devices which give a representation of a multivariate distribution function in terms of its univariate marginal distributions. Copulas explain the dependence structure among variables through the second approach above by eliminating the effect of univariate marginals. A historical review and major developments of copulas can be found in Sklar [14], Schweizer and Sklar [13], Dall’Aglio [3], Schweizer [12], Kotz [7], Nelsen [10] and Drouet Mari and Kotz [4] among others.

Rodríguez-Lallena and Úbeda-Flores [11] proposed a new class of bivariate copulas depending on two univariate functions, which is a generalization of Farlie-Gumbel-Morgenstern (FGM) families of copulas. However, their class of bivariate copulas is based only on a specific case of copulas and thus, has a limitation on extending their results to broader cases of copulas because of using only independent copula. This motivates the authors to examine more generalized cases that includes Rodríguez-Lallena and Úbeda-Flores (RLUF)$^\dagger$ [11] bivariate copulas.

In this paper, we incorporate the proposed new bivariate copulas into various iterated copula families and investigate several properties of the new class of copulas, such as dependence properties and measures of association between two random variables. Also, we introduce the regression function for the new class of copulas and provide several practical examples using the bivariate iterated copulas.

This paper is organized as follows: Section 2 contains a description of the copulas. Section 3 introduces our new class of bivariate iterated copulas. The properties of the new class of copulas are explored in Section 4. Directional dependence using copula is addressed in Section 5. Conclusions in Section 6 summarize the results and future research directions.

\*Address for correspondence: Jong-Min Kim, E-mail: jongminkim@morris.umn.edu.
2. Definitions and preliminaries

In this section, we summarize the basic results that are necessary to understand the concept of a copula. We focus on some preliminary properties of a copula and present some standard bivariate copula families. A copula is a multivariate distribution function defined on \([0, 1]^n\), with uniformly distributed marginals. In this paper, we focus on a bivariate (two-dimensional) copula, where \(n = 2\).

Definition 1 A bivariate copula is a function \(C : [0, 1]^2 \rightarrow [0, 1]\), whose domain is the entire unit square with the following three properties:

(i) \(C(u, 0) = C(0, v) = 0\), \(\forall u, v \in [0, 1]\)
(ii) \(C(u, 1) = C(1, u) = u\), \(\forall u \in [0, 1]\)
(iii) \(C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0\), \(\forall u_1, u_2, v_1, v_2 \in [0, 1]\) such that \(u_1 \leq u_2\) and \(v_1 \leq v_2\).

Sklar [15] shows that any bivariate distribution function \((F_{XY})\), can be represented as a function of its marginal distribution of \(X\) and \(Y\) \((F_X\) and \(F_Y\)) by using a two-dimensional copulas \(C(\cdot, \cdot)\). More specifically, a copula may be written as

\[F_{XY}(x, y) = C(F_X(x), F_Y(y)).\]

If \(F_X\) and \(F_Y\) are continuous, then copula, \(C\), is unique and invariant under strictly monotone transformations of the random variables. Let \(X\) and \(Y\) be continuously distributed random variables with copula \(C\), and let \(h_1\) and \(h_2\) be strictly increasing transformation functions. Then, the copula of \(h_1(x)\) and \(h_2(y)\) is the same as the copula of \(x\) and \(y\), called the invariance property, (i.e., \(C(h_1(x), h_2(y)) = C(x, y)\)). Therefore, if \(F_{XY}(x, y)\) is a joint distribution function with continuous marginals, \(F_X(x) = u\) and \(F_Y(y) = v\), then \(F_{XY}(x, y)\) can be represented in terms of a unique copula function as follows:

\[F_{XY}(x, y) = C(F_X(x), F_Y(y)) = C(u, v),\]

where \(C(u, v)\) is the copula of \(F_{XY}(x, y)\). Therefore, the copula function represents how the multivariate function \(F_{XY}(x, y)\) is constructed from its marginal distribution functions, \(F_X(x)\) and \(F_Y(y)\). Also a copula describes the dependence mechanism between two random variables by eliminating the influence of the marginals or any monotone transformation on the marginals. See Nelson [10] for further systematic exposition and references therein.

3. New class of bivariate copulas

One of the parametric family of copulas is the Farlie-Gumbel-Morgenstern(FGM) family defined when \(\theta \in [-1, 1]\) by

\[C_{\alpha}^{FGM}(u, v) = uv + \theta uv(1 - u)(1 - v), \forall u, v \in [0, 1],\]

and studied in Farlie [5], Gumbel [6], and Morgenstern [9]. A well known limitation to this family is that it does not allow the modeling of large dependencies since Spearman’s Rho is limited to \(\rho \in [-1/3, 1/3]\). Rodríguez-Lallena and Úbeda-Flores [11] suggests a wide class of bivariate copulas that depends on two univariate function and describes the dependency of the copulas by

\[C^{RLLF}(u, v) = uv + f(u)g(v), \forall u, v \in [0, 1].\]

We tried to generalize the class of bivariate copulas suggested by Rodríguez-Lallena and Úbeda-Flores [11] by

\[C(u, v) = C^*(u, v) + f(u)g(v), \forall u, v \in [0, 1],\]

where \(C^*\) can be any copula. But if \(f\) and \(g\) satisfy the conditions from Theorem 2.3 in Rodríguez-Lallena and Úbeda-Flores [11] then \(C\) in (3) is a valid copula. Therefore, our objective of this paper is partially accomplished. Theorem 1 below, states that \(C\) in (3) is a bivariate copula under suitable conditions.
Theorem 1 Let \( f \) and \( g \) be two non-zero real functions defined on \([0, 1]\). Let \( C = [0, 1]^2 \to \mathbb{R} \) be the function defined by (1). Suppose that \( f \) and \( g \) satisfy the following conditions
(a) \( f(0) = f(1) = g(0) = g(1) = 0 \),
(b) \( f \) and \( g \) are absolutely continuous,
(c) \( \min\{\alpha \delta, \beta \delta\} \geq -\frac{(u_2 - v_1)(u_2 - v_2)}{(v_2 - v_1)} \), where \( \forall u_1, u_2, v_1, v_2 \in [0, 1] \) such that \( u_1 \leq u_2, v_1 \leq v_2, \eta = C^*(u_1, v_1) - C^*(u_2, v_1) + C^*(u_2, v_2) \), \( \alpha = \inf\{f'(u) : u \in A\} < 0, \beta = \sup\{f'(u) : u \in A\} > 0, \gamma = \inf\{g'(v) : v \in B\} < 0, \) and \( \delta = \sup\{g'(v) : v \in B\} > 0 \), with \( A = \{u \in [0, 1] : f'(u) \text{ exists}\} \) and \( B = \{v \in [0, 1] : g'(v) \text{ exists}\} \).

Then \( C \) is a copula.

Note that if \( C \) follows the condition (b) in Theorem 1, then \( C \) is absolutely continuous.

Proof. The proof is straightforward from Lemmas 2.1-2 and Theorem 2.3 in Rodríguez-Lallena and Úbeda-Flores [11].

In Theorem 1, we extend the condition (c), which was first introduced in Theorem 2.3 in Rodríguez-Lallena and Úbeda-Flores [11], into a general form of appropriate bivariate function. By incorporating \( C^* \) into the form of \( C \), our proposed class of copula contains that of Rodríguez-Lallena and Úbeda-Flores [11]. For example, if we consider \( C^*(u, v) \) as \( uv \), our class of copula reduces to that of Rodríguez-Lallena and Úbeda-Flores [11]. In addition, it is easy to see that the bivariate functions considered by Lai and Xie [8] and Bairamov and Kotz [2] are the special case of our new class of bivariate copulas. Recently, Amblard and Girard [1] proposed a new family of copulas generalizing the FGM family.

The following corollary provides that our proposed class of copulas also contains many new parametric families, which include all the cases of Rodríguez-Lallena and Úbeda-Flores [11]. The proof of this corollary is also obvious.

Let \( f \) and \( g \) be two non-zero absolutely continuous functions defined on \([0, 1]\) such that \( f(0) = f(1) = g(0) = g(1) = 0 \). Let \( C_\theta \) be the function defined on \([0, 1]^2\) by \( C_\theta(u, v) = C^*(u, v) + \theta f(u)g(v), \) where \( \theta \in \mathbb{R} \). If there exists a following bound
\[
(\frac{\eta}{(u_2 - u_1)(v_2 - v_1)} - \frac{\eta}{(u_2 - u_1)(v_2 - v_1)}) \max\{\alpha \delta, \beta \delta\} \leq \theta \leq -\frac{(u_2 - u_1)(v_2 - v_1)}{(v_2 - v_1)} \min\{\alpha \delta, \beta \delta\},
\]
where \( \forall u_1, u_2, v_1, v_2 \in [0, 1] \) such that \( u_1 \leq u_2, v_1 \leq v_2, \) then \( C_\theta \) is a copula.

The preceding results lead to the following theorem below, which gives a form of iterated bivariate copulas based on the convex combination of products of two appropriate marginal functions. Theorem 2 tells us that the proposed class of copulas contains many commonly used copulas. In Section 5, we provide a few of examples of copulas that fall into the proposed class and has also been studied for the class of copulas introduced in Rodríguez-Lallena and Úbeda-Flores’ [11].

Theorem 2 Let \( f_1, \ldots, f_m \) and \( g_1, \ldots, g_n \) be functions on \([0, 1]\) such that their derivatives are bounded and their integrals are zero over \([0, 1]\). Also, let \( C^* \) be a copula. Define \( C(u, v) = C^*(u, v) + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(u)g_j(v) \) under the conditions in Theorem 1. Then there exist parameters \( \xi_{ij} \) such that \( C \) is a copula.

Proof. In order to show that \( C \) is a copula, we need to show that there exist \( \xi_{ij} \) such that the three conditions for copula have been satisfied.

(i) \( C(w, 0) = C(0, w) = 0 \): Since,
\[
C(w, 0) = C^*(w, 0) + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(w)g_j(0) = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(w)g_j(0) = 0,
\]
\[
C(0, w) = C^*(0, w) + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(0)g_j(w) = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(0)g_j(w) = 0,
\]
(ii) \( C(w, 1) = C(1, w) = 0 \): Since,
\[ C(w, 1) = C^*(w, 1) + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(w) g_j(1) = w + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(w) g_j(1) = w, \]

\[ C(1, w) = C^*(1, w) + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(1) g_j(w) = w + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(1) g_j(w) = w, \]

where
\[ \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(1) g_j(w) = 0. \]

(iii) 2-increasing property:
This is equivalent to the condition that copula density function, \( c(u, v) \), should be nonnegative,
\[ c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v} = c^*(u, v) + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i'(u) g_j'(v) \geq 0. \]

Since \( f_i' \) and \( g_j' \) are bounded and \( \inf \{ c^*(u, v) \} > 0 \), the parameters \( \xi_{ij} \) can be chosen such that \( c \geq 0 \).

4. Properties of the new class of copulas

For better understanding of the dependence structure in s copula, nonparametric measures of association such as Spearsman’s \( \rho \) and Kendall’s \( \tau \) are quite useful.

For a joint distribution function \( F_X Y(x, y) \), its corresponding copula is given by
\[ C(u, v) = C(F_X^{-1}(u), F_Y^{-1}(y)), \]
where \( F_X^{-1}(u) = \inf \{ x : F_X(x) = u \} \) (i.e., \( F_X^{-1} \) is the inverse function of \( F_X \)), similarly for \( F_Y^{-1}(v) \). The Farlie-Gumbel-Morgenstern (FGM) bivariate distribution \([5]\) has copula as follows:
\[ C_{\alpha}^{FGM}(u, v) = uv[1 + \alpha (1 - u)(1 - v)], \quad \alpha \in [-1, 1], \]
which is a special case of our new class of copulas \( C^*_\psi(u, v) = C^*(u, v) + \psi f(u) g(v) \).

FGM copula does not allow the modeling of large dependencies. To permit the modeling of high positive dependence, Bairamov and Kotz [2] extended the FGM class of bivariate distributions to increase the dependence between the components and proposed some properties of the FGM class of bivariate distributions. Our new class of bivariate copulas is the generalization of the FGM bivariate distributions, and, thus, we incorporate those properties given by Bairamov and Kotz [2] into our new class of bivariate copulas. In this section, we study some of the properties of this new class of bivariate copulas.

**Theorem 3** Suppose that \( f \) and \( g \) are defined in Theorem 3.3. Let \((U^*, V^*)\) be a pair of random variables with copula \( C^* \) and uniform marginals. Similarly, let \((U, V)\) be a pair of uniform random variables with copula \( C^*_\psi(u, v) = C^*(u, v) + \psi f(u) g(v) \) under the conditions in Theorem 1. Then,
\[ f(v) = g(v) \text{ if and only if } P\{U < V\} = P\{U^* < V^*\}. \]

**Proof.** (i) To show that \( f(v) = g(v) \Rightarrow P\{U < V\} = P\{U^* < V^*\} \), let
\[ c^*_\psi(u, v) = \frac{\partial^2 C^*_\psi(u, v)}{\partial u \partial v} = c^*(u, v) + \psi f'(u) g'(v) \geq 0. \]

Then, 

\[ \frac{\partial^2 C^*_\psi}{\partial u \partial v} = c^*(u, v) + \psi f'(u) g'(v) \geq 0. \]
\[ P(U < V) = \int_0^1 \int_0^v [c^*(u, v) + \psi f'(u)g'(v)] \, du \, dv \]
\[ = \int_0^1 \int_0^v c^*(u, v) \, du \, dv + \int_0^v \psi \int_0^1 f'(u)g'(v) \, du \, dv \]
\[ = P(U^* < V^*) + \psi \int_0^1 f(v)g'(v) \, dv. \quad (4) \]

Since \( f(v) = g(v) \), and \( f(0) = f(1) = g(0) = g(1) = 0 \),
\[ \int_0^1 f(v)g'(v) \, dv = f(1)g(1) - f(0)g(0) - \int_0^1 f'(v)g(v) \, dv = - \int_0^1 f(v)g'(v) \, dv. \]

Therefore, \( \int_0^1 f(v)g'(v) \, dv = 0 \) in (8) becomes zero.
We showed that \( f(v) = g(v) \) implies \( P(U < V) = P(U^* < V^*) \).

(ii) To show that \( P(U < V) = P(U^* < V^*) \) \( \Rightarrow \) \( f(v) = g(v) \). We can show that (ii) is true by performing the same procedure as we showed in (i).

Therefore, \( f(v) = g(v) \) if and only if \( P(U < V) = P(U^* < V^*) \).

For example, suppose we have \( C_u(u, v) = C^*(u, v)[1 + \alpha(1 - u)(1 - v)] \), \( \alpha \in [-1, 1] \), and \( C^*(u, v) = uv \).

Then,
\[ P(U < V) = \int_0^1 \int_0^v \frac{\partial}{\partial u} \left[ C_u(u, v) \right] \, du \, dv = \int_0^1 \int_0^v [1 + \alpha(1 - 2u)(1 - 2v)] \, du \, dv \]
\[ = \frac{1}{2} + \int_0^1 \int_0^v [\alpha(1 - 2u)(1 - 2v)] \, du \, dv = \frac{1}{2} \]

\[ P(U^* < V^*) = \int_0^1 \int_0^v \frac{\partial}{\partial v} \left[ C^*(u, v) \right] \, du \, dv = \int_0^1 v \, dv = \frac{1}{2} \]

It means that \( P(U < V) = P(U^* < V^*) \), where \( f(v) = g(v) = v(1 - v) \). Consider now \( E(U^*|V^* \leq a) \), where \( \{V^* \leq a\} \) is the right-sided marginal truncation region for \( V^* \). The conditional distribution of \( U^* \) under the condition \( \{V^* \leq a\} \) is
\[ P(U^* \leq v|V^* \leq a) = \frac{P(U^* \leq v, V^* \leq a)}{P(V^* \leq a)} = \frac{C^*(u, a)}{a}. \]

Obviously, the conditional density function of \( U^* \) under the condition \( \{V^* \leq a\} \) is
\[ f_{U^*|V^* \leq a}(u) = \frac{1}{a} \frac{\partial}{\partial u} [C^*(u, a)]. \]

The conditional expectation of \( U^* \) can be written as
\[ E(U^*|V^* \leq a) = \frac{1}{a} \int_0^1 u \frac{\partial}{\partial u} [C^*(u, a)] \, du = \frac{1}{a} \{ 1 - \int_0^1 [C^*(u, a)] \, du \} \]

Let \( \zeta \) be a class of copulas of the form \( C(u, v) = C^*(u, v) + f(u)g(v) \). Let us apply the preceding result to our new class \( \zeta \) and then derive the simple form as follows:
\[ E(U|V \leq a) = \frac{1}{a} \{ 1 - \int_0^1 [C(u, a)] \, du \} = \frac{1}{a} \{ 1 - \int_0^1 [C^*(u, a) + f(u)g(a)] \, du \} \]
\[ = \frac{1}{a} \{ 1 - \int_0^1 C^*(u, a) \, du - \int_0^1 f(u)g(a) \, du \} \]
\[
\frac{1}{a} \left\{ 1 - \int_0^1 C^*(u, a) du \right\} - \frac{1}{a} \int_0^1 f(u) g(a) du \\
\quad = E(U^*|V^* = a) - \frac{g(a)}{a} \int_0^1 f(u) du. 
\]

Now we know we consider the measure of association in this new class of bivariate copulas.

From Nelsen [10], we can get the following property: let \( X \) and \( Y \) be continuous random variables with a copula \( C \). Then Kendall’s \( \tau \) is given by

\[
\tau_C = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1, \tag{5}
\]

where \( U \) and \( V \) are two uniform random variables in \((0, 1)\).

**Theorem 4** Suppose that \( f \) and \( g \) are defined in Theorem 3.3 and both \( C \) and \( C^* \) are copulas, and let \( C(u, v) = C^*(u, v) + \lambda(u, v) \) where \( \lambda(u, v) = f(u)g(v) \), then we can derive the following Kendall’s \( \tau \):

\[
\tau_{\lambda, \lambda} = \tau_{C^*} + 4 \int_0^1 \int_0^1 \lambda(u, v) d\lambda(u, v) \\
\quad + 4 \left( \int_0^1 \int_0^1 C^*(u, v) f'(u)g'(v) dudv + \int_0^1 \int_0^1 \lambda(u, v) \frac{\partial^2 C^*(u, v)}{\partial u \partial v} dudv \right). \tag{6}
\]

**Proof.** Using Eq. (5), we derive the following:

\[
\tau_C = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1 \\
\quad = 4 \int_0^1 \int_0^1 (C^*(u, v) + \lambda(u, v)) d(C^*(u, v) + \lambda(u, v)) - 1 \\
\quad = 4 \int_0^1 \int_0^1 C^*(u, v) dC^*(u, v) + 4 \int_0^1 \int_0^1 C^*(u, v) d\lambda(u, v) \\
\quad + 4 \int_0^1 \int_0^1 \lambda(u, v) dC^*(u, v) + 4 \int_0^1 \int_0^1 \lambda(u, v) d\lambda(u, v) - 1 \\
\quad = \tau_{C^*} + 4 \int_0^1 \int_0^1 \lambda(u, v) d\lambda(u, v) \\
\quad + 4 \left( \int_0^1 \int_0^1 C^*(u, v) d\lambda(u, v) + \int_0^1 \int_0^1 \lambda(u, v) dC^*(u, v) \right), \\
\quad = \tau_{C^*} + 4 \int_0^1 \int_0^1 \lambda(u, v) d\lambda(u, v) \\
\quad + 4 \left( \int_0^1 \int_0^1 C^*(u, v) f'(u)g'(v) dudv + \int_0^1 \int_0^1 \lambda(u, v) \frac{\partial^2 C^*(u, v)}{\partial u \partial v} dudv \right),
\]

which is proved.

Lai and Xie [8] describe a new family of positive quadrant dependent (PQD) bivariate distribution and consider the modified FGM copula of the form

\[
C^{LX}(u, v) = uv\{1 + \alpha(1-u)^p(1-v)^p\}, \quad p \geq 1, 0 \leq \alpha \leq 1,
\]

which can be expressed with our proposed new class of bivariate copula,
\[ C_\psi(u, v) = C^*(u, v) + \psi f(u)g(v). \]

If we replace \( C^*(u, v) = uv, \psi = \alpha, f(u) = u(1-u)^p \), and \( g(v) = v(1-v)^p \) in our proposed class, we obtain the same modified FGM copula suggested by Lai and Xie [8]. Using Theorem 4.2., we derive the Kendall’s \( \tau \) of \( C_\psi(u, v) \).

\[
\tau_{C_\psi} = \tau_C + 4 \int_0^1 \int_0^1 \lambda(u, v) d\lambda(u, v) \\
+ 4 \left( \int_0^1 \int_0^1 C^*(u, v) f'(u)g' dudv + \int_0^1 \int_0^1 \lambda(u, v) \frac{\partial^2 C^*(u, v)}{\partial u \partial v} dudv \right).
\]

Since the Kendall’s \( \tau \) of \( C^*(u, v) = uv \) is zero,

\[
\int_0^1 \int_0^1 \lambda(u, v) d\lambda(u, v) = \int_0^1 \int_0^1 \alpha f(u)g(v) d\left( \alpha f(u)g(v) \right) \\
= \int_0^1 \int_0^1 \alpha u(1-u)^p v(1-v)^p d\left( \alpha u(1-u)^p v(1-v)^p \right) \\
= \alpha^2 \int_0^1 u(1-u)^p d\left( u(1-u)^p \right) \int_0^1 v(1-v)^p d\left( v(1-v)^p \right) \\
= \alpha^2 \left[ \int_0^1 u(1-u)^2 du - \int_0^1 u^2(1-u)^{2p-1} du \right] \times \\
\left[ \int_0^1 v(1-v)^2 dv - \int_0^1 v^2(1-v)^{2p-1} dv \right] \\
= \alpha^2 \left[ B(2, 2p + 1) - pB(3, 2p) \right] \left[ B(2, 2p + 1) - pB(3, 2p) \right] \\
= \alpha^2 \left[ B(2, 2p + 1) - pB(3, 2p) \right]^2.
\]

Similarly,

\[
\int_0^1 \int_0^1 C^*(u, v)f'(u)g' dudv = \alpha \int_0^1 \int_0^1 uv \left( (1-u)^p - pu(1-u)^{p-1} \right) \\
\times \left( (1-v)^p - pv(1-v)^{p-1} \right) dudv \\
= \alpha \left[ B(2, p + 1)^2 - 2pB(2, p + 1)B(3, p) + p^2 B(3, p)^2 \right],
\]

and

\[
\int_0^1 \int_0^1 \lambda(u, v) \frac{\partial^2 C^*(u, v)}{\partial u \partial v} dudv = \alpha \int_0^1 \int_0^1 u(1-u)^p v(1-v)^p dudv \\
= \alpha B(2, p + 1)^2.
\]

So we can derive the Kendall’s \( \tau \) of \( C_\psi(u, v) \) as follows:

\[
\tau_{C_\psi} = 4\alpha^2 \left[ B(2, 2p + 1) - pB(3, 2p) \right]^2 + 4\alpha \left[ 2B(2, p + 1)^2 - 2pB(2, p + 1)B(3, p) + p^2 B(3, p)^2 \right],
\]

where \( B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \) and \( \Gamma(\cdot) \) is the well known gamma function.
5. Examples

In Theorem 2, we considered a broad class of copulas, and in this section, we provide some examples by constructing new bivariate copulas based on Theorem 2 in the following steps.

1. Start with a copula \( C, f_1, \ldots, f_m \) and \( g_1, \ldots, g_m \) be functions on \([0,1]\) such that their derivatives are bounded and their integrals are zero over \([0,1]\).
2. Find the conditions on the parameters \( \xi_{ij} \) so that

   \[
   \begin{align*}
   (i) \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(w)g_j(0) = 0, \\
   (ii) \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(0)g_j(w) = 0, \\
   (iii) \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(u)g_j(1) = w, \\
   (iv) \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i(1)g_j(w) = w, \\
   (v) \quad & c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v} = c^*(u,v) + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} f_i'(u)g_j'(v) \geq 0.
   \end{align*}
   \]

Now, we provide three examples that satisfy the form of a copula suggested in Theorem 2. Note that those copulas in the following examples are the special cases of Theorem 2 and are contained in the class of copulas in Rodríguez-Lallena and Úbeda-Flores \[11\]. This also implies that the proposed class of copulas is more generalization than that of Rodríguez-Lallena and Úbeda-Flores \[11\]. Before providing examples, we want to introduce a directional dependence as follows:

**Theorem 5** Let \((U^*, V^*)\) and \((U, V)\) be 2 pairs of uniform random variables with copulas \( C^* \) and \( C \in \zeta \), respectively. Then we obtain the following:

\[
\begin{align*}
(i) \quad \text{Corr}(U, V) &= \rho_C = \rho_{C^*} + 12 \int_0^1 f(u)du \int_0^1 g(v)dv, \\
(ii) \quad E(V|U = u) &= 1 - \int_0^1 [C_u(v)]dv = E(V^*|U^* = u) - f'(u) \int_0^1 g(v)dv,
\end{align*}
\]

where \( \text{Corr}(U, V) \) denotes Spearman's correlation and

\[
C_u(v) = P(V \leq v|U = u) = \frac{\partial C(u,v)}{\partial u}.
\]

**Proof.** (i) The proof directly follows from the definition of Spearman's correlation as follows:

\[
\rho_C = 12 \int_0^1 \int_0^1 [C(u,v) - uv]dudv.
\]

\[
\text{Corr}(U, V) = 12 \int_0^1 \int_0^1 [C(u,v) - uv]dudv = 12 \int_0^1 \int_0^1 [C^*(u,v) + f(u)g(v) - uv]dudv
\]

\[
= 12 \int_0^1 \int_0^1 [C^*(u,v) - uv]dudv + \int_0^1 \int_0^1 f(u)g(v)dudv
\]

\[
= \text{Corr}(U^*, V^*) + 12 \int_0^1 f(u)du \int_0^1 g(v)dv.
\]
(ii) Note that \( P(V \leq v | U = u) = \frac{\partial C(u,v)}{\partial u} = C_v(u) = C^*_u(v) + f^*(u)g(v), \)

We can easily show that

\[
E(V | U = u) = 1 - \int_0^1 [C_u(v)]dv = 1 - \int_0^1 [C^*_u(v)]dv - f^*(u) \int_0^1 g(v)dv
\]

\[
= E(V^* | U^* = u) - f^*(u) \int_0^1 g(v)dv
\]

where \( C_u(v) = P(V \leq v | U = u) = \frac{\partial C(u,v)}{\partial u}. \)

It is interesting to note that

\[
E(V | U = u) = E(V^* | U^* = u) - f^*(u)\left\{ \frac{12 \int_0^1 f(u)du \int_0^1 g(v)dv}{12 \int_0^1 f(u)du} \right\}
\]

\[
= E(V^* | U^* = u) - f^*(u)\left\{ \frac{\rho_C - \rho^*_C}{12 \int_0^1 f(u)du} \right\},
\]

where \( \rho_C = \rho_{C^*} + 12 \int_0^1 f(u)du \int_0^1 g(v)dv. \)

**Example 1.** Let \( C^*(u, v) \) and \( f_1(w) = g_1(w) = w(w - 1) \). For \( m = n = 1 \), (i) · (v) will not impose any condition on the parameter \( \xi_{11} \), since \( f_1(0) = g_1(1) = g_1(0) = g_1(1) = 0 \). The condition (v) implies that

\[
1 + \xi_{11}(2u - 1)(2v - 1) \geq 0 \Rightarrow \xi_{11} \geq -\frac{1}{(2u - 1)(2v - 1)} \Rightarrow |\xi_{11}| \leq 1.
\]

This result will lead to a copula \( C(u, v) = uv[1 + \xi_{11}(u - 1)(v - 1)] \) that is well known as FGM class. A more generalized version of FGM class by using Theorem 3.3 is

\[
C(u, v) = uv[1 + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij}(u^i - 1)(v^j - 1)].
\]

The regression function for this class can be expressed as

\[
E(V | U = u) = 1 - \int_0^1 C_u(v)dv
\]

\[
= 1 - \int_0^1 \left[ v + u \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij}(u^i - 1)(v^j - 1) + u \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij}u^i(v^j - 1) \right] dv
\]

\[
= 1 - \left[ \frac{1}{2} - \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} \left( \frac{j}{2(j + 2)} \right)(u^i - 1) - \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} \left( \frac{j}{2(j + 2)} \right)u^i \right]
\]

\[
= \left[ \frac{1}{2} + \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} \left( \frac{j}{2(j + 2)} \right) ((i + 1)u^i - 1) \right].
\]

**Example 2.** Let \( C^*(u, v), f_1(w) = g_1(w) = \frac{1 - e^{\lambda u}}{1 - e^\lambda} - \frac{1 - e^{\gamma v}}{1 - e^\gamma} \). Then, the new class of copula will have the following form as:

\[
C(u, v) = uv + \xi_{11} \left[ \left( \frac{1 - e^{\lambda u}}{1 - e^\lambda} - \frac{1 - e^\gamma v}{1 - e^\gamma} \right) \left( \frac{1 - e^{\lambda u}}{1 - e^\lambda} - \frac{1 - e^{\gamma v}}{1 - e^\gamma} \right) \right], \quad \lambda, \gamma > 0.
\]

The regression function for this class can be expressed as
\[ E(V|U = u) = 1 - \int_0^1 C_u(v) dv \]
\[ = 1 - \int_0^1 \left[ v + \xi_{11} \left( \frac{\lambda e^{-\lambda u} - \gamma e^{-\gamma u}}{1 - e^{-\lambda}} - \frac{\gamma e^{-\gamma u}}{1 - e^{-\gamma}} \right) \right] dv \]
\[ = \frac{1}{2} - \xi_{11} \left( \frac{\lambda e^{-\lambda u} - \gamma e^{-\gamma u}}{1 - e^{-\lambda}} - \frac{\gamma e^{-\gamma u}}{1 - e^{-\gamma}} \right) \int_0^1 \left( \frac{1 - e^{-\lambda x}}{1 - e^{-\lambda}} - \frac{1 - e^{-\gamma x}}{1 - e^{-\gamma}} \right) dx \]
\[ = \frac{1}{2} - \xi_{11} \left( \frac{1}{1 - e^{-\lambda}} - \frac{1}{1 - e^{-\gamma}} + \frac{1}{\gamma} \right) \left( \frac{\lambda e^{-\lambda u} - \gamma e^{-\gamma u}}{1 - e^{-\lambda}} - \frac{\gamma e^{-\gamma u}}{1 - e^{-\gamma}} \right). \]

**Example 3.** One way of finding functions \( f \) and \( g \) that satisfy the conditions (a) and (b) given in Theorem 3.3 is the following. Let \( F_1, F_2, G_1 \) and \( G_2 \) be distribution functions of nonnegative random variables. Define \( f(u) = \frac{F_1(u)}{F_2(u)} - \frac{F_2(u)}{F_2(1)} \) and \( g(v) = \frac{G_1(v)}{G_2(v)} - \frac{G_2(v)}{G_2(1)} \). Then, for any \( C^*(u, v) \), we may construct a new class of copulas as follows:

\[
C(u, v) = C^*(u, v) + \left[ \frac{F_1(u)}{F_1(1)} - \frac{F_2(u)}{F_2(1)} \right] \left[ \frac{G_1(v)}{G_1(1)} - \frac{G_2(v)}{G_2(1)} \right]
\]

6. **Discussion and further remark**

Dependence properties and measures of association between two or more variables can be investigated in terms of various copulas. In this paper, we have presented a flexible class of bivariate iterated copulas with interesting and useful properties and dependence structures that extends the idea of Rodríguez-Lallena and Úbeda-Flores [11]. We have also provided several properties and examples that follow the proposed procedure.

Sungur [18] has proposed a directional dependence in regression setting based on Rodríguez-Lallena and Úbeda-Flores [11] class of bivariate copulas. We refer readers to Sungur [18] for more detail. However, this topic does require further investigation. Therefore, our new class of bivariate copulas will be useful when researchers study the concept of directional dependence in bivariate regression setting by using copulas. In addition, it is possible to build a new class of multivariate copulas using the truncation invariant dependence structure presented by Sungur [16,17].

**References**